

# CSE 311 Section 6

More Proofs, Induction

# Administrivia



# Announcements & Reminders

- HW4 due yesterday!
- HW5
  - due 11/05 @ 11:59 pm



# Proof by Contrapositive



# Proof by Contrapositive

If we are trying to prove  $p \rightarrow q$ , it might be easier to prove the contrapositive

- $p \rightarrow q$  is equivalent to  $\neg q \rightarrow \neg p$

## example proof template

*We will **argue by contrapositive** and show if not  $q$ , then not  $p$*

*Suppose [not  $q$ ]*

*✨...magic... ✨*

*Thus we have shown not  $p$*

*Therefore our claim is true!*

## The *Trifecta* (Question 2)

Consider the following proposition:

For each integer  $a$ , if 3 divides  $a^2$ , then 3 divides  $a$ .

(a) Write the contrapositive of this claim

(b) Prove this claim using a proof by contrapositive

*Hint: Think about cases based on the possible values after mod by 3*

Work on this with the people around you and then we'll go over it together!

## The *Trifecta* (Question 2)

Consider the following proposition:

For each integer  $a$ , if **3 divides  $a^2$** , then **3 divides  $a$** .

(a) Write the contrapositive of this claim

For each integer  $a$ , if **3 does not divide  $a$** , then **3 does not divide  $a^2$**

*Remember that the contrapositive is **equivalent** to the original expression!*

## The *Trifecta* (Question 2)

For each integer  $a$ , if 3 does not divide  $a$ , then *3 does not divide  $a^2$*

**(b) Prove this claim using a proof by contrapositive**

We will **argue by contrapositive**. Let  $a$  be an arbitrary integer.

We will show that if 3 does not divide  $a$ , then 3 does not divide  $a^2$

Suppose 3 does not divide  $a$ .

✨....magic.... ✨

Thus we have shown that 3 does not divide  $a^2$

Since  $a$  was arbitrary, and we proved the contrapositive, the claim holds!

## The *Trifecta* (Question 2)

For each integer  $a$ , if 3 does not divide  $a$ , then *3 does not divide  $a^2$*

**(b) Prove this claim using a proof by contrapositive**

We will **argue by contrapositive**. Let  $a$  be an arbitrary integer.

We will show that if 3 does not divide  $a$ , then 3 does not divide  $a^2$

We're making a claim about ALL integers, so we introduce an arbitrary variable

Suppose 3 does not divide  $a$ .

✨....magic.... ✨

Thus we have shown that 3 does not divide  $a^2$

Since  $a$  was arbitrary, and we proved the contrapositive, the claim holds!

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We will **argue by contrapositive**. Let  $a$  be an arbitrary integer.

We will show that if 3 does not divide  $a$ , then 3 does not divide  $a^2$

Suppose 3 does not divide  $a$ .

This means that  $a$  must have a remainder of either 1 or 2 when dividing by 3.

**Case 1:**  $a \equiv 1 \pmod{3}$

**Case 2:**  $a \equiv 2 \pmod{3}$

Thus we have shown that 3 does not divide  $a^2$

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Suppose 3 does not divide  $a$ .

This means that  $a$  must have a remainder of either 1 or 2 when dividing by 3.

**Case 1:**  $a \equiv 1 \pmod{3}$

$a$  can be expressed as an integer with remainder 1 as:  $a = 3k + 1, k \in \mathbb{Z}$

**Case 2:**  $a \equiv 2 \pmod{3}$

$a$  can be expressed as an integer with remainder 2 as:  $a = 3k + 2, k \in \mathbb{Z}$

Thus we have shown that 3 does not divide  $a^2$

Since  $a$  was arbitrary, and we proved the contrapositive, the claim holds!

We want to make a claim about  $a^2$ ,  
so let's compute it!

## The *Trifecta* (Question 2)

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**Case 1:**  $a \equiv 1 \pmod{3}$

$a$  can be expressed as an integer with remainder 1 as:  $a = 3k + 1$ ,  $k \in \mathbb{Z}$

Then,  $a^2$  as  $a \cdot a = (3k + 1) \cdot (3k + 1) = 9k^2 + 6k + 1 = 3(3k^2 + 2k + 1/3)$

**Case 2:**  $a \equiv 2 \pmod{3}$

$a$  can be expressed as an integer with remainder 2 as:  $a = 3k + 2$ ,  $k \in \mathbb{Z}$

Then,  $a^2$  as  $a \cdot a = (3k + 2) \cdot (3k + 2) = 9k^2 + 12k + 4 = 9k^2 + 12k + 3 + 1 = 3(3k^2 + 4k + 1 + 1/3)$

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By closure under add and multiplies,  $3k^2 + 2k$  is an integer, but adding  $1/3$  makes this a fraction!

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Thus we have shown that 3 does not divide  $a^2$

Since  $a$  was arbitrary, and we proved the contrapositive, the claim holds!

# Proof by Contradiction



# Proof by Contradiction

If we are trying to prove  $p$ , we may show it's impossible for  $\neg p$  to be true

## example proof template

*For the sake of contradiction suppose not  $p$*

*This means that ...*

*✨...magic... ✨*

*But we have found a CONTRADICTION!!!*

*Therefore  $p$  must be true*

# Prime Checking (Question 4)

We will use “nontrivial divisor” to mean a factor that isn’t 1 or the number itself.

Formally, a positive integer  $k$  being a “nontrivial divisor” of  $n$  means that  $k|n$ ,  $k \neq 1$  and  $k \neq n$ .

**Claim:** For every positive integer  $n$ , if  $n$  has a nontrivial divisor, then it has a nontrivial divisor at most  $\sqrt{n}$ .

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**Claim:** For every positive integer  $n$ , if  $n$  has a nontrivial divisor, then it has a nontrivial divisor at most  $\sqrt{n}$ .

Let's start by taking the *negation* of this claim.

1. Write in predicate logic:

Let  $D(n, k)$  be true iff  $k$  is a nontrivial divisor of  $n$ .

$$\forall n(\exists k(D(n, k)) \rightarrow \exists k(D(n, k) \wedge k \leq \sqrt{n}))$$

2. Take the negation:

$$\exists n[\exists k(D(n, k)) \wedge \forall k(D(n, k) \rightarrow (k > \sqrt{n}))]$$

3. Convert back to English:

**There is an  $n$  such that  $n$  has a non-trivial divisor and all its nontrivial divisors are greater than  $\sqrt{n}$**

## Prime Checking (Question 4)

Suppose for the sake of contradiction that **there is an  $n$  such that  $n$  has a non-trivial divisor and all its nontrivial divisors are greater than  $\sqrt{n}$**

This means that ...

✨...magic... ✨

But we have found a CONTRADICTION!!!

Therefore it must be true that **for every positive integer  $n$ , if  $n$  has a nontrivial divisor, then it has a nontrivial divisor at most  $\sqrt{n}$ .**

*Work on filling in the “magic” with the people around you!*

Start by writing down things we know...

## Prime Checking (Question 4)

Suppose for the sake of contradiction that **there is an  $n$  such that  $n$  has a non-trivial divisor and all its nontrivial divisors are greater than  $\sqrt{n}$**

**Let  $k$  be a nontrivial divisor of  $n$ . Since  $k$  is a divisor,  $n = kc$  for some integer  $c$ .**

???

But we have found a CONTRADICTION!!!

Therefore it must be true that **for every positive integer  $n$ , if  $n$  has a nontrivial divisor, then it has a nontrivial divisor at most  $\sqrt{n}$ .**

and making "random" observation...

## Prime Checking (Question 4)

Suppose for the sake of contradiction that **there is an  $n$  such that  $n$  has a non-trivial divisor and all its nontrivial divisors are greater than  $\sqrt{n}$**

Let  $k$  be a nontrivial divisor of  $n$ . Since  $k$  is a divisor,  $n = kc$  for some integer  $c$ .

**Observe that  $c$  is also a divisor  $c|n$  since there is a integer  $k$  such that  $n = ck$ .**

???

But we have found a CONTRADICTION!!!

Therefore it must be true that **for every positive integer  $n$ , if  $n$  has a nontrivial divisor, then it has a nontrivial divisor at most  $\sqrt{n}$ .**

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Therefore it must be true that **for every positive integer  $n$ , if  $n$  has a nontrivial divisor, then it has a nontrivial divisor at most  $\sqrt{n}$ .**

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**Since both  $k$  and  $c$  are non-trivial divisors, we have that  $k > \sqrt{n}$  and  $c > \sqrt{n}$ .**

**Thus,  $kc > \sqrt{n} * \sqrt{n} = n$**

???

But we have found a CONTRADICTION!!!

Therefore it must be true that **for every positive integer  $n$ , if  $n$  has a nontrivial divisor, then it has a nontrivial divisor at most  $\sqrt{n}$ .**

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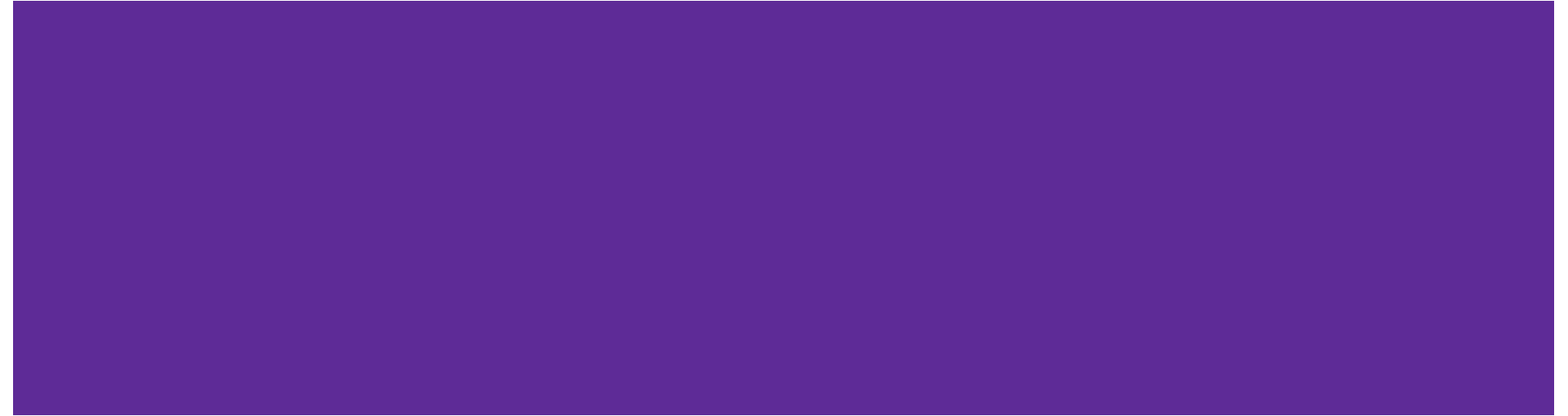
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**But we have  $kc = n$ , and  $kc$  can't also be  $> n$  - we have found a CONTRADICTION!!!**

Therefore it must be true that **for every positive integer  $n$ , if  $n$  has a nontrivial divisor, then it has a nontrivial divisor at most  $\sqrt{n}$ .**

# Set Practice



# How Many Elements? (Question 5)

For each of these, how many elements are in the set?

$$(a) A = \{1, 2, 3, 2\}$$

$$(b) B = \{\emptyset, \{\emptyset\}, \{\emptyset, \emptyset\}, \{\emptyset, \emptyset, \emptyset\}, \dots\}$$

$$(c) C = A \times (B \cup \{7\})$$

# How Many Elements? (Question 5)

For each of these, how many elements are in the set?

(a)  $A = \{1, 2, 3, 2\}$

$|A| = 3$

Sets only care about the distinct elements: 1, 2, 3

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2

3

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(b)  $B = \{\emptyset, \{\emptyset\}, \{\emptyset, \emptyset\}, \{\emptyset, \emptyset, \emptyset\}, \dots\}$



There are all the same as  $\{\emptyset\}$  because duplicates don't count!

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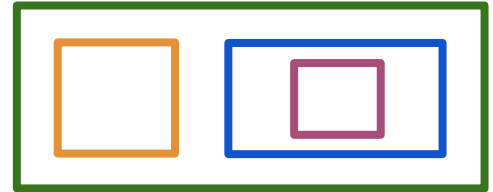
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(b)  $B = \{\emptyset, \{\emptyset\}, \{\emptyset, \emptyset\}, \{\emptyset, \emptyset, \emptyset\}, \dots\}$

$|B| = 2$

$B = \{\emptyset, \{\emptyset, \emptyset, \emptyset, \dots\}\} = \{\emptyset, \{\emptyset\}\}$  – 2 elements!



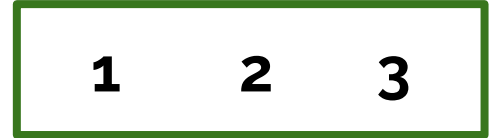
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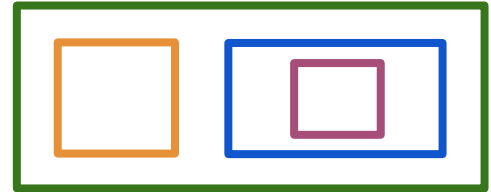
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$$(b) B = \{\emptyset, \{\emptyset\}, \{\emptyset, \emptyset\}, \{\emptyset, \emptyset, \emptyset\}, \dots\}$$

$$|B| = 2$$

$B = \{\emptyset, \{\emptyset\}, \{\emptyset, \emptyset\}, \{\emptyset, \emptyset, \emptyset\}, \dots\} = \{\emptyset, \{\emptyset\}\}$  – 2 elements!



$$(c) C = A \times (B \cup \{7\})$$

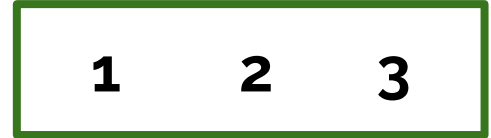
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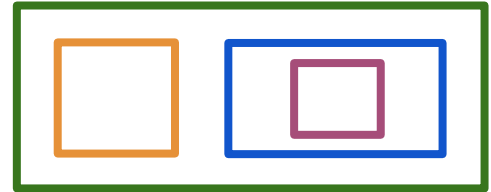
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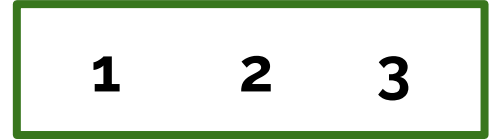
this union has  $2 + 1 = 3$  elements

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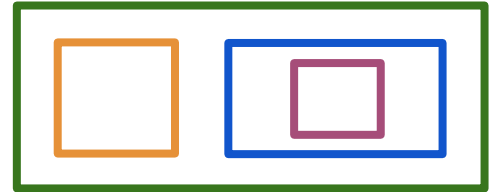
$$(a) A = \{1, 2, 3, 2\} \quad |A| = 3$$

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$$(b) B = \{\emptyset, \{\emptyset\}, \{\emptyset, \emptyset\}, \{\emptyset, \emptyset, \emptyset\}, \dots\} \quad |B| = 2$$

$B = \{\emptyset, \{\emptyset\}, \{\emptyset, \emptyset\}, \{\emptyset, \emptyset, \emptyset\}, \dots\} = \{\emptyset, \{\emptyset\}\}$  – 2 elements!



$$(c) C = A \times (B \cup \{7\}) \quad |C| = 9$$

$C = \{1, 2, 3\} \times \{\emptyset, \{\emptyset\}, 7\} = \{(a, b) \mid a \in \{1, 2, 3\}, b \in \{\emptyset, \{\emptyset\}, 7\}\}$

So, there are  $3 * 3 = 9$  total pairs that we will form

# Set Proofs

Set proofs are like regular proofs we've done before, with some new notation!  
*for example...*

- **Subsets:** If you want to prove  $A \subseteq B$ 
  - We want to show any *arbitrary* element we find in A must also be in B
  - Formally, we are showing  $\forall x(x \in A \rightarrow x \in B)$ 
    - This can be done with a regular direct proof!
- **Equality:** If you want to prove  $A = B$ ....
  - We want to show any *arbitrary* element we find in B must also be in A AND any arbitrary element we find in A must also be in B
  - Formally we are showing  $\forall x(x \in A \rightarrow x \in B \wedge x \in B \rightarrow x \in A)$
- **Inequality, not a subset, etc.:** Take the negation and prove as normal!
  - e.g.,  $A \not\subseteq B$  is  $\neg \forall x(x \in A \rightarrow x \in B) \equiv \exists x(x \in A \wedge x \notin B)$ 
    - intuitively, A isn't a subset of B if there's an element in A that's not in B

You can also still use the proof techniques we've done before like proof by contradiction, contrapositive, cases, etc.

## Set Proof (Question 8b)

Prove that  $(A \cap B) \times C \subseteq A \times (C \cup D)$  for any sets A, B, C, D.

*Work on this proof with the people around you and then we'll go over it together!*

## Set Proof (Question 8b)

Prove that  $(A \cap B) \times C \subseteq A \times (C \cup D)$  for any sets  $A, B, C, D$ .

Let  $x$  be an arbitrary element of  $(A \cap B) \times C$ .

✨MAGIC ✨

Thus,  $x$  is also an element of  $A \times (C \cup D)$ .

Since  $x$  was arbitrary, by definition of subset, we have shown  $(A \cap B) \times C \subseteq A \times (C \cup D)$ .

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Let  $x$  be an arbitrary element of  $(A \cap B) \times C$ .

**By definition of Cartesian product,  $x$  is of form  $(y, z)$  where  $y \in A \cap B$  and  $z \in C$**

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**By definition of intersection,  $y \in A$  AND  $y \in B$ ; in particular, note that  $y \in A$ .**

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By definition of intersection,  $y \in A$  AND  $y \in B$ ; in particular, note that  $y \in A$ .

**Since  $z \in C$ , by definition of  $\cup$ , we also have  $z \in C \cup D$ .**

✨MAGIC ✨

Thus,  $x$  is also an element of  $A \times (C \cup D)$ .

Since  $x$  was arbitrary, by definition of subset, we have shown  $(A \cap B) \times C \subseteq A \times (C \cup D)$ .

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Since  $z \in C$ , by definition of  $\cup$ , we also have  $z \in C \cup D$ .

**Therefore since  $y \in A$  and  $z \in C \cup D$ , by definition of Cartesian product we have  $x = (y, z) \in A \times (C \cup D)$ .**

Thus,  $x$  is also an element of  $A \times (C \cup D)$ .

Since  $x$  was arbitrary, by definition of subset, we have shown  $(A \cap B) \times C \subseteq A \times (C \cup D)$

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Prove  $A \cap \overline{B} \subseteq A \setminus B$  for any sets  $A, B$ .

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✨MAGIC ✨

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By definition of intersection,  $x \in A$  AND  $x \in \overline{B}$ .

**By definition of complement,  $x \notin B$**

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**Since,  $x \in A$  AND  $x \notin B$ , by definition of set difference,  $x$  is an element of  $A \setminus B$ .**

Since  $x$  was arbitrary, by definition of subset, we have shown  $A \cap \overline{B} \subseteq A \setminus B$

# Induction



# (Weak) Induction Template

Let  $P(n)$  be “(whatever you’re trying to prove)”.

We show  $P(n)$  holds for all  $n$  by induction on  $n$ .

Base Case: Show  $P(b)$  is true.

Inductive Hypothesis: Suppose  $P(k)$  holds for an arbitrary  $k \geq b$ .

Inductive Step: Show  $P(k + 1)$  (i.e. get  $P(k) \rightarrow P(k + 1)$ )

Conclusion: Therefore,  $P(n)$  holds for all  $n$  by the principle of induction.

# (Weak) Induction Template

Let  $P(n)$  be “(whatever you’re trying to prove)”.  
We show  $P(n)$  holds **for all  $n$**  by induction on  $n$ .

Note: often you will condition  $n$  here, like “all natural numbers  $n$ ” or “ $n \geq 0$ ”

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Conclusion: Therefore,  $P(n)$  holds **for all  $n$**  by the principle of induction.

Match the earlier condition on  $n$  in your conclusion!

# Problem 10 – Induction with Equality

- a) Show using induction that  $0 + 1 + 2 + \dots + n = \frac{n(n+1)}{2}$  for all  $n \in \mathbb{N}$ .
- b) Define the triangle numbers as  $\Delta_n = 1 + 2 + \dots + n$ , where  $n \in \mathbb{N}$ . In part (a) we showed  $\Delta_n = \frac{n(n+1)}{2}$ . Prove the following equality for all  $n \in \mathbb{N}$  :
- $$0^3 + 1^3 + \dots + n^3 = \Delta_n^2$$

Lets walk through part (a) together.

We can “fill in” our induction template to construct our proof by induction.

# Problem 10 – Induction with Equality

Show using induction that

$$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for all  $n \in \mathbb{N}$ .

Let  $P(n)$  be “”. We show  $P(n)$  holds for (some)  $n$  by induction on  $n$ .

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a) Show using induction that  $0 + 1 + 2 + \dots + n = \frac{n(n+1)}{2}$  for all  $n \in \mathbb{N}$ .

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$$0^3 + 1^3 + \dots + n^3 = \Delta_n^2$$

Now try part (b) with people around you, and then we'll go over it together!

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$$\Delta_n = 1 + 2 + \cdots + n, \quad n \in \mathbb{N}.$$

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$$= \left(\frac{k(k+1)}{2}\right)^2 + (k + 1)^3 \quad \text{by (a)}$$

$$= (k + 1)^2 \left(\frac{k^2}{2^2} + (k + 1)\right) \quad \text{factor out } (k + 1)^2$$

$$= (k + 1)^2 \left(\frac{k^2 + 4k + 4}{4}\right)$$

$$= (k + 1)^2 \left(\frac{(k+2)^2}{4}\right) \quad \text{factor numerator}$$

$$= \left(\frac{(k+1)(k+2)}{2}\right)^2$$

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Conclusion: Therefore,  $P(n)$  holds for all  $n \in \mathbb{N}$  by the principle of induction.

# **That's All, Folks!**

**Thanks for coming to section this week!  
Any questions?**