weak/regular

strong

structural

base case ≠ 0

multiple base cases

Even More Induction

CSE 311 Autumn 2025
Lecture 20

Recursively Defined Functions: Fibonacci

Just like induction works will with recursive code, it also works well for recursively-defined functions.

1, 1, 2, 3, 5, 8, …

Define the Fibonacci numbers as follows:

$f(0) = 1$

$f(1) = 1$

$f(n) = f(n-1) + f(n-2)$ for all $n \in \mathbb{N}, n \geq 2$.

*This is a somewhat unusual definition, $f(0) = 0, f(1) = 1$ is more common.

Fibonacci Inequality: goal

Show that $f(n) \leq 2^n$ for all $n \geq 0$ by induction.

(1) Define P(n)

(2) Base Case(s)

(3) IH

(4) IS

(5) Conclusion

$$f(0) = 1; \qquad f(1) = 1$$

$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Fibonacci Inequality: skeleton

Show that $f(n) \leq 2^n$ for all $n \geq 0$ by induction.

Define $P(n)$ to be "$f(n) \leq 2^n$" We show $P(n)$ is true for all $n \geq 0$ by induction on n.

Base Cases: $(n = 0)$: $f(0) = 1 \leq 1 = 2^0$.

$(n = 1)$: $f(1) = 1 \leq 2 = 2^1$.

$P(0)$ ✓

$P(1)$

Inductive Hypothesis: Suppose $P(0) \wedge P(1) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq 1$.

$f(k) \leq 2^k$

Inductive step: $f(k+1) = f(k) + f(k-1) \leq 2^k + 2^{k-1}$

$2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

Target: $P(k+1)$. i.e. $f(k+1) \leq 2^{k+1}$

$$f(0) = 1; \qquad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Fibonacci Inequality

Show that $f(n) \leq 2^n$ for all $n \geq 0$ by induction.

Define $P(n)$ to be "$f(n) \leq 2^n$" We show $P(n)$ is true for all $n \geq 0$ by induction on n.

Base Cases: $(n = 0)$: $f(0) = 1 \leq 1 = 2^0$.

$(n = 1)$: $f(1) = 1 \leq 2 = 2^1$.

Inductive Hypothesis: Suppose $P(0) \wedge P(1) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq 1$.

Inductive step: $f(k+1) = f(k) + f(k-1)$ by the definition of the Fibonacci numbers. Applying IH twice, we have $f(k+1) \leq 2^k + 2^{k-1} < 2^k + 2^k = 2^{k+1}$.

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

$$f(0) = 1; \qquad f(1) = 1$$

$f(n) = f(n-1) + f(n-2)$ for all $n \in \mathbb{N}, n \geq 2$.

Induction: Hats! (goal)

You have n people in a line $(n \geq 2)$. Each of them wears either a purple hat or a gold hat. The person at the front of the line wears a purple hat. The person at the back of the line wears a gold hat.

Show that for every arrangement of the line satisfying the rule above, there is a person with a purple hat next to someone with a gold hat.

Yes this is kinda obvious. I promise this is good induction practice.

Yes you could argue this by contradiction. I promise this is good induction practice.

What is $P(n)$?

Induction: Hats! ($P(n)$)

Define $P(n)$ to be "in every line of n people with gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show $P(n)$ for all integers $n \geq 2$ by induction on n.

Base Case: $n = 2$

Inductive Hypothesis:

Inductive Step:

By the principle of induction, we have $P(n)$ for all $n \geq 2$

Induction: Hats! (base case, IH)

String p q
cases

Define $P(n)$ to be "in every line of n people with gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show $P(n)$ for all integers $n \geq 2$ by induction on n.

Base Case: $n = 2$ The line must be just a person with a purple hat and a person with a gold hat, who are next to each other.

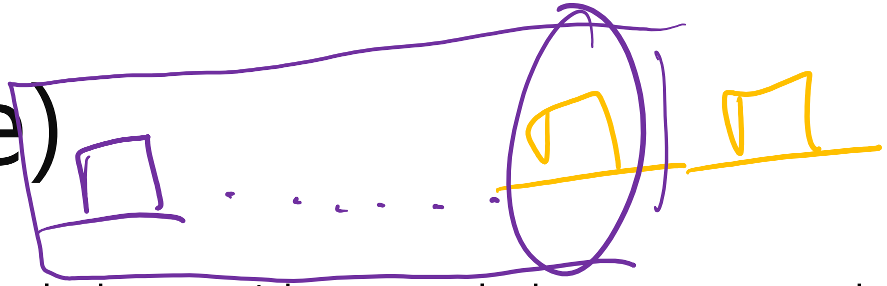
Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step: Consider an arbitrary line with $k + 1$ people in purple and gold hats, with a gold hat at one end and a purple hat at the other.

Target: there is someone in a purple hat next to someone in a gold hat.

By the principle of induction, we have $P(n)$ for all $n \geq 2$

Induction: Hats! (complete)



Define $P(n)$ to be "in every line of n people with gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show $P(n)$ for all integers $n \geq 2$ by induction on n.

Base Case: $n = 2$ The line must be just a person with a purple hat and a person with a gold hat, who are next to each other.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step: Consider an arbitrary line with $k + 1$ people in purple and gold hats, with a gold hat at one end and a purple hat at the other.

Case 1: There is someone with a purple hat next to the person in the gold hat at one end. Then those people are the required adjacent opposite hats.

Case 2:. There is a person with a gold hat next to the person in the gold hat at the end. Then the line from the second person to the end is length k, has a gold hat at one end and a purple hat at the other. Applying the inductive hypothesis, there is an adjacent, opposite-hat wearing pair.

In either case we have $P(k + 1)$.

By the principle of induction, we have $P(n)$ for all $n \geq 2$

Structural Induction on Strings

Strings

ε is "the empty string"

The string with 0 characters – `""` in Java (not `null`!)

Σ^*:

Basis: $\varepsilon \in \Sigma^*$.

Recursive: If $w \in \Sigma^*$ and $a \in \Sigma$ then $wa \in \Sigma^*$

wa means the string of w with the character a appended.

You'll also see $w \cdot a$ (a $\cdot$ to mean "concatenate" i.e. + in Java)

Functions on Strings

Since strings are defined recursively, most functions on strings are as well.

Length:

len(ε)=0;

len(wa)=len(w)+1 for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R = \varepsilon$;
$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation

$x \cdot \varepsilon = x$ for all $x \in \Sigma^*$;
$x \cdot (wa) = (x \cdot w)a$ for $w \in \Sigma^*, a \in \Sigma$

Number of c's in a string

$\#_c(\varepsilon) = 0$
$\#_c(wc) = \#_c(w) + 1$ for $w \in \Sigma^*$;
$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*, a \in \Sigma \setminus \{c\}$.

String proof:
Claim for all $x, y \in \Sigma^*$ len(x·y)=len(x) + len(y).

$\forall x \forall y$

Let $P(y)$ be "for all $x \in \Sigma^*$ len(x·y)=len(x) + len(y)."

Notice the strangeness of this $P()$ there is a "for all x" inside the definition of $P(y)$.

That means we'll have to introduce an arbitrary x as part of the base case and the inductive step!

String proof: skeleton

len(x∘ε) = len(x) + len(ε) $\forall x \in \Sigma^*$

Let $P(y)$ be "len(x·y)=len(x) + len(y) for all $x \in \Sigma^*$."

We prove $P(y)$ for all $x \in \Sigma^*$ by structural induction.

Base Case: $P(\varepsilon)$ x arb string in Σ^*

len(x∘ε) = len(x)

= len(x) + 0

= len(x) + len(ε)

Inductive Hypothesis

Inductive Step:

We conclude that $P(y)$ holds for all string y by the principle of induction. Unwrapping the definition of P, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

String proof: Base case, IH

Let $P(y)$ be "len(x·y)=len(x) + len(y) for all $x \in \Sigma^*$."

We prove $P(y)$ for all $x \in \Sigma^*$ by structural induction.

Base Case: Let x be an arbitrary string, len($x \cdot \epsilon$)=len(x) =len(x)+0=len(x)+len(ε)

Inductive Hypothesis: Suppose $P(w)$ for an arbitrary string w.

Inductive Step:

We conclude that $P(y)$ holds for all string y by the principle of induction. Unwrapping the definition of P, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

P(wa)

String proof: Inductive Step Setup

Let $P(y)$ be "len(x·y)=len(x) + len(y) for all $x \in \Sigma^*$."

We prove $P(y)$ for all $x \in \Sigma^*$ by structural induction.

Base Case: Let x be an arbitrary string, len($x \cdot \epsilon$)=len(x) =len(x)+0=len(x)+len(ε)

Inductive Hypothesis: Suppose $P(w)$ for an arbitrary string w.

Inductive Step: Let $y = wa$ for an arbitrary $a \in \Sigma$. We show $P(y)$. Let x be an arbitrary string.

…

Therefore, len(xy)=len(x) + len(y), as required.

We conclude that $P(y)$ holds for all string y by the principle of induction. Unwrapping the definition of P, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

Goal $\forall x \in \Sigma^*$

len(x·y) = len(x·wa)

= len(x) + len(y)

String proof: complete

Let $P(y)$ be "len(x·y)=len(x) + len(y) for all $x \in \Sigma^*$."

We prove $P(y)$ for all $x \in \Sigma^*$ by structural induction.

Base Case: Let x be an arbitrary string, len($x \cdot \epsilon$)=len(x) =len(x)+0=len(x)+len(ε)

Inductive Hypothesis: Suppose $P(w)$ for an arbitrary string w.

Inductive Step: Let $y = wa$ for an arbitrary $a \in \Sigma$. We show $P(y)$. Let x be an arbitrary string.

len(xy)=len(xwa) =len(xw)+1 (by definition of len)

=len(x) + len(w) + 1 (by IH)

=len(x) + len(wa) (by definition of len)

Therefore, len(xy)=len(x) + len(y), as required.

We conclude that $P(y)$ holds for all string y by the principle of induction. Unwrapping the definition of P, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

Why all those arbitraries?

Let $P(y)$ be "len(x·y)=len(x) + len(y) for all $x \in \Sigma^*$."

We prove $P(y)$ for all $x \in \Sigma^*$ by structural induction.

Base Case: Let x be an arbitrary string, len($x \cdot \epsilon$)=len(x) =len(x)+0=len(x)+len(ε)

Inductive Hypothesis: Suppose $P(w)$ for an arbitrary string w.

Inductive Step: Let $y = wa$ for an arbitrary $a \in \Sigma$. We show $P(y)$. Let x be an arbitrary string.

len(xy)=len(xwa) =len(xw)+1 (by definition of len)

=len(x) + len(w) + 1 (by IH)

=len(x) + len(wa) (by definition of len)

Therefore, len(xy)=len(x) + len(y), as required.

We conclude that $P(y)$ holds for all strings y by the principle of induction. Unwrapping the definition of P, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

$P(\varepsilon)$ is a for-all statement, introduce arbitrary variable to show for-all.

Needs to be arbitrary because it's in the IH (induction wouldn't show "all strings" otherwise)

Recursive rule says "every $a \in \Sigma$" so we need to argue for every a.

$P(y)$ is a for-all statement, introduce arbitrary variable to show for-all.

More Practice

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$. (structure)

[Define $P(n)$]

Base Case

Inductive Hypothesis

Inductive Step

[conclusion]

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$. (1)

Let $P(n)$ be "$3|(2^{2n}-1)$." We show $P(n)$ holds for all $n \in \mathbb{N}$.

Base Case ($n = 0$) note that $2^{2n} - 1 = 2^0 - 1 = 0$. Since $3 \cdot 0 = 0$, and 0 is an integer, $3|(2^{2\cdot 0}-1)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step:

Target: $P(k + 1)$, i.e. $3|(2^{2(k+1)}-1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$. (2)

Let $P(n)$ be "$3|(2^{2n}-1)$." We show $P(n)$ holds for all $n \in \mathbb{N}$.

Base Case ($n = 0$) note that $2^{2n} - 1 = 2^0 - 1 = 0$. Since $3 \cdot 0 = 0$, and 0 is an integer, $3|(2^{2\cdot 0}-1)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step: By inductive hypothesis, $3|(2^{2k}-1)$. i.e. there is an integer j such that $3j = 2^{2k} - 1$.

$2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1$

FORCE the expression in your IH to appear

Target: $P(k+1)$, i.e. $3|(2^{2(k+1)}-1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$. (3)

Let $P(n)$ be "$3|(2^{2n}-1)$." We show $P(n)$ holds for all $n \in \mathbb{N}$.

Base Case ($n = 0$) note that $2^{2n} - 1 = 2^0 - 1 = 0$. Since $3 \cdot 0 = 0$, and 0 is an integer, $3|(2^{2\cdot 0}-1)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step: By inductive hypothesis, $3|(2^{2k}-1)$. i.e. there is an integer j such that $3j = 2^{2k} - 1$.

$2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1 = 4\left(2^{2k} - 1\right) + 4 - 1$

By IH, we can replace $2^{2k} - 1$ with $3j$ for an integer j

$2^{2(k+1)} - 1 = 4(3j) + 4 - 1 = 3(4j) + 3 = 3(4j + 1)$

Since $4j + 1$ is an integer, we meet the definition of divides and we have:

Target: $P(k + 1)$, i.e. $3|(2^{2(k+1)}-1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$. (intuition)

That inductive step might still seem like magic.

It sometimes helps to run through examples, and look for patterns:

$2^{2\cdot 0} - 1 = 0 = 3 \cdot 0$

$2^{2\cdot 1} - 1 = 3 = 3 \cdot 1$

$2^{2\cdot 2} - 1 = 15 = 3 \cdot 5$

$2^{2\cdot 3} - 1 = 63 = 3 \cdot 21$

$2^{2\cdot 4} - 1 = 255 = 3 \cdot 85$

$2^{2\cdot 5} - 1 = 1023 = 3 \cdot 341$

The divisor goes from k to $4k + 1$

$0 \to 4 \cdot 0 + 1 = 1$

$1 \to 4 \cdot 1 + 1 = 5$

$5 \to 4 \cdot 5 + 1 = 21$

…

That might give us a hint that $4k + 1$ will be in the algebra somewhere, and give us another intermediate target.

Fibonacci Inequality Two (structure)

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

[Define $P(n)$]

Base Cases:

Inductive Hypothesis:

Inductive step:

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

$$f(0) = 1; \qquad f(1) = 1$$

$f(n) = f(n-1) + f(n-2)$ for all $n \in \mathbb{N}, n \geq 2$.

Fibonacci Inequality Two (1)

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

Define $P(n)$ to be "$f(n) \geq 2^{n/2}$" We show $P(n)$ is true for all $n \geq 2$ by induction on n.

Base Cases: $f(2) = f(1) + f(0) = 2 \geq 2 = 2^1 = 2^{2/2}$

$f(3) = f(2) + f(1) = 2 + 1 = 3 = 2 \cdot \frac{3}{2} \geq 2\sqrt{2} = 2^{1.5} = 2^{3/2}$

Inductive Hypothesis: Suppose $P(2) \wedge P(3) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq 3$.

Inductive step: $f(k+1) = f(k) + f(k-1)$ by the definition of the Fibonacci numbers. Applying IH twice, we have

Target: $f(k+1) \geq 2^{(k+1)/2}$

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

$$f(0) = 1; \qquad f(1) = 1$$
$f(n) = f(n-1) + f(n-2)$ for all $n \in \mathbb{N}, n \geq 2$.

Fibonacci Inequality Two (2)

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

Define $P(n)$ to be "$f(n) \geq 2^{n/2}$" We show $P(n)$ is true for all $n \geq 2$ by induction on n.

Base Cases: $f(2) = f(1) + f(0) = 2 \geq 2 = 2^1 = 2^{2/2}$

$f(3) = f(2) + f(1) = 2 + 1 = 3 = 2 \cdot \frac{3}{2} \geq 2\sqrt{2} = 2^{1.5} = 2^{3/2}$

Inductive Hypothesis: Suppose $P(2) \wedge P(3) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq 3$.

Inductive step: $f(k+1) = f(k) + f(k-1)$ by the definition of the Fibonacci numbers. Applying IH twice, we have

$f(k+1) \geq 2^{k/2} + 2^{(k-1)/2}$

$\geq 2^{(k+1)/2}$

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

$$f(0) = 1; \qquad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Fibonacci Inequality Two (3)

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

Define $P(n)$ to be "$f(n) \geq 2^{n/2}$" We show $P(n)$ is true for all $n \geq 2$ by induction on n.

Base Cases: $f(2) = f(1) + f(0) = 2 \geq 2 = 2^1 = 2^{2/2}$

$f(3) = f(2) + f(1) = 2 + 1 = 3 = 2 \cdot \frac{3}{2} \geq 2\sqrt{2} = 2^{1.5} = 2^{3/2}$

Inductive Hypothesis: Suppose $P(2) \wedge P(3) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq 3$.

Inductive step: $f(k+1) = f(k) + f(k-1)$ by the definition of the Fibonacci numbers. Applying IH twice, we have

$$
\begin{aligned}
f(k+1) &\geq 2^{k/2} + 2^{(k-1)/2} \\
&= 2^{(k-1)/2}\left(\sqrt{2} + 1\right) \\
&\geq 2^{(k-1)/2} \cdot 2 \\
&\geq 2^{(k+1)/2}
\end{aligned}
$$

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

$$f(0) = 1; \qquad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Even More Induction Practice: g(n) and h(n) (1)

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$

Claim: $h(n) \geq g(n)$ for all integers $n \geq 1$

Even More Induction Practice: g(n) and h(n) (2)

Define $P(n)$ to be "$\mathrm{h}(n) \geq g(n)$ for all integers $n \geq 1$

We show $P(n)$ for all $n \geq 1$ by induction on n.

Base Case

Inductive Hypothesis:

Inductive Step:

Thus $P(k+1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$

Even More Induction Practice: g(n) and h(n) (3)

Define $P(n)$ to be "$\mathrm{h}(n) \geq g(n)$ for all integers $n \geq 1$

We show $P(n)$ for all $n \geq 1$ by induction on n.

Base Case ($n = 1$): $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$.

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$g(k+1) = (k+1) \cdot g(k)$

$= (k+1)^{k+1}$.

Thus $P(k+1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$

Even More Induction Practice: g(n) and h(n) (4)

Define $P(n)$ to be "$\mathrm{h}(n) \geq g(n)$ for all integers $n \geq 1$

We show $P(n)$ for all $n \geq 1$ by induction on n.

Base Case $(n = 1)$: $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$.

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$$g(k+1) = (k+1) \cdot g(k)$$
$$\leq (k+1) \cdot h(k) \text{ by IH.}$$

$$= (k+1)^{k+1}.$$

Thus $P(k+1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$

Even More Induction Practice: g(n) and h(n) (5)

Define $P(n)$ to be "$\mathrm{h}(n) \geq g(n)$ for all integers $n \geq 1$

We show $P(n)$ for all $n \geq 1$ by induction on n.

Base Case ($n = 1$): $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$.

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$$
\begin{aligned}
g(k+1) &= (k+1) \cdot g(k) \\
&\leq (k+1) \cdot h(k) && \text{by IH.} \\
&\leq (k+1) \cdot k^k && \text{by definition of } h(k) \\
&\leq (k+1) \cdot (k+1)^k \\
&= (k+1)^{k+1}.
\end{aligned}
$$

Thus $P(k+1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$

Even More Induction Practice: g(n) and h(n) (6)

Define $P(n)$ to be "$\mathrm{h}(n) \geq g(n)$ for all integers $n \geq 1$

We show $P(n)$ for all $n \geq 1$ by induction on n.

Base Case ($n = 1$): $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$.

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$$\begin{aligned} g(k+1) &= (k+1) \cdot g(k) \\ &\leq (k+1) \cdot h(k) && \text{by IH.} \\ &\leq (k+1) \cdot k^k && \text{by definition of } h(k) \\ &\leq (k+1) \cdot (k+1)^k \\ &= (k+1)^{k+1}. \end{aligned}$$

Thus $P(k+1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$

Even More Induction Practice: Sums (1)

Let $P(n)$ be $\sum_{i=0}^{n} 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on n.

Base Case ($n = 0$):

Inductive Hypothesis:

Inductive Step:

[Conclusion]

Even More Induction Practice: Sums (2)

Let $P(n)$ be $\sum_{i=0}^{n} 2+3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on n.

Base Case ($n = 0$): $\sum_{i=0}^{0} 2+3i = 2 = \frac{4}{2} = \frac{(0+1)(3\cdot 0+4)}{2}$

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 0$.

Inductive Step:

Target: $\sum_{i=0}^{k+1} 2+3i = \frac{([k+1]+1)(3[k+1]+4)}{2}$

Even More Induction Practice: Sums (3)

Let $P(n)$ be $\sum_{i=0}^{n} 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on n.

Base Case ($n = 0$): $\sum_{i=0}^{0} 2 + 3i = 2 = \frac{4}{2} = \frac{(0+1)(3 \cdot 0+4)}{2}$

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 0$.

Inductive Step:

$\sum_{i=0}^{k+1} 2 + 3i = (\sum_{i=0}^{k} 2 + 3i) + \left(2 + 3(k+1)\right)$. By IH, we have:

$\sum_{i=0}^{k+1} 2 + 3i = \frac{(\mathrm{k}+1)(3\mathrm{k}+4)}{2} + 2 + 3\mathrm{k} + 3 = ????$

$$= \frac{([\mathrm{k}+1]+1)(3[\mathrm{k}+1]+4)}{2}$$

Even More Induction Practice: Sums (4)

Let $P(n)$ be $\sum_{i=0}^{n} 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on n.

Base Case ($n = 0$): $\sum_{i=0}^{0} 2 + 3i = 2 = \frac{4}{2} = \frac{(0+1)(3\cdot 0+4)}{2}$

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 0$.

Inductive Step:

$\sum_{i=0}^{k+1} 2 + 3i = (\sum_{i=0}^{k} 2 + 3i) + \big(2 + 3(k+1)\big)$. By IH, we have:

$\sum_{i=0}^{k+1} 2 + 3i = \frac{(\mathrm{k}+1)(3\mathrm{k}+4)}{2} + 2 + 3\mathrm{k} + 3 = \frac{3\mathrm{k}^2+7\mathrm{k}+4}{2} + \frac{6\mathrm{k}+10}{2} = \frac{3\mathrm{k}^2+13\mathrm{k}+14}{2} = \frac{(3\mathrm{k}+7)(\mathrm{k}+2)}{2} = \frac{([\mathrm{k}+1]+1)(3[\mathrm{k}+1]+4)}{2}$

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.