Warm up:
What is the following recursively-defined set?
Basis Step: $4 \in S$, $5 \in S$
Recursive Step: If $x \in S$ and $y \in S$ then $x - y \in S$

$x = 4, y = 4$
$x = 5, y = 4$

$S = \{\ldots, -1, 0, 1, 2, 3, 4, 5, \ldots\}$
$= \mathbb{Z}.$

Structural Induction and Regular Expressions

CSE 311 Autumn 2025
Lecture 19

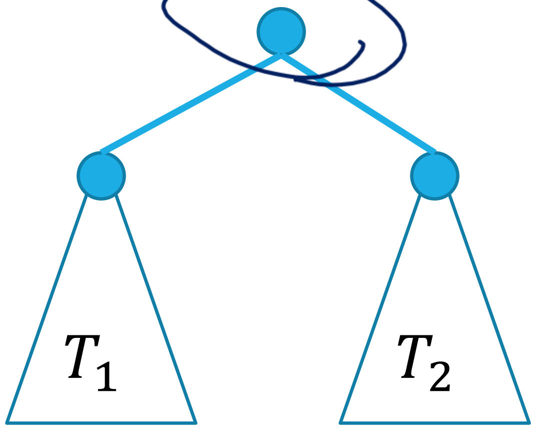
Trees!

More Structural Sets

Binary Trees are another common source of structural induction.

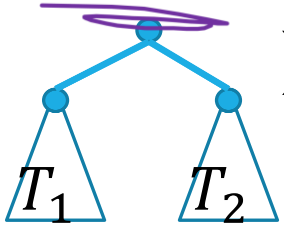
Basis: A single node is a rooted binary tree.

Recursive Step: If T_1 and T_2 are rooted binary trees with roots r_1 and r_2, then a tree rooted at a new node, with children r_1, r_2 is a binary tree.



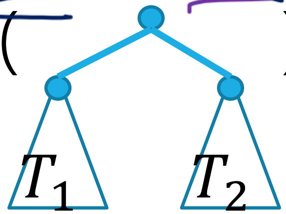
Functions on Binary Trees

size(●)=1

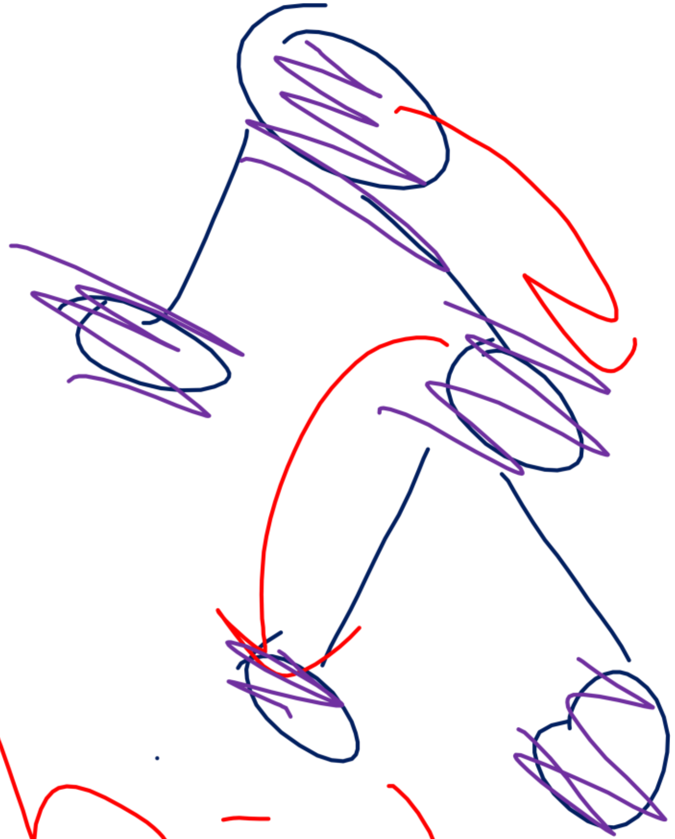
size(

) = size(T_1) + size(T_2) + 1

height(●) = 0

height(

) = 1+max(height(T_1),height(T_2))



Claim

We want to show that trees of a certain height can't have too many nodes. Specifically our claim is this:

For all trees T, size(T) $\leq 2^{height(T)+1} - 1$

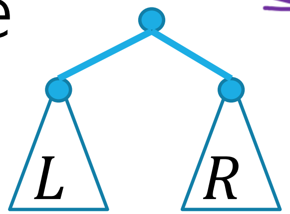
Take a moment to absorb this formula, then we'll do induction!

Structural Induction on Binary Trees

Let $P(T)$ be "size(T) $\leq 2^{height(T)+1} - 1$". We show $P(T)$ for all binary trees T by structural induction.

Base Case: Let $T =$ ●. size(T)=1 and height(T) = 0, so size(T)=1$\leq 2 - 1 = 2^{0+1} - 1 = 2^{height(T)+1} - 1$.

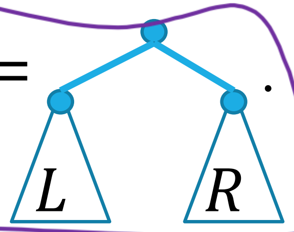
Inductive Hypothesis: Suppose $\mathrm{P}(L)$ and $\mathrm{P}(R)$ hold for arbitrary trees L, R. Let T be the tree



Inductive step: Figure out, (1) what we must show (2) a formula for height and a formula for size of T.

Structural Induction on Binary Trees (cont.)

Let $P(T)$ be "size$(T) \leq 2^{height(T)+1} - 1$". We show $P(T)$ for all binary trees T by structural induction.

$T =$ 

.

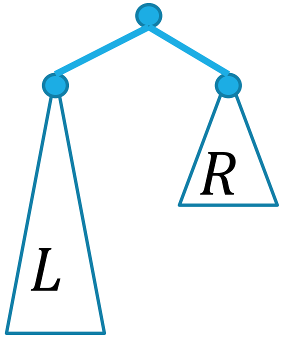
height(T)=$1 + \max\{height(L), height(R)\}$

size(T)= 1 +size(L)+size(R)

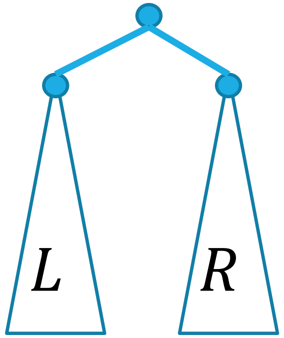
So $P(T)$ holds, and we have $P(T)$ for all binary trees T by the principle of induction.

How do heights compare?

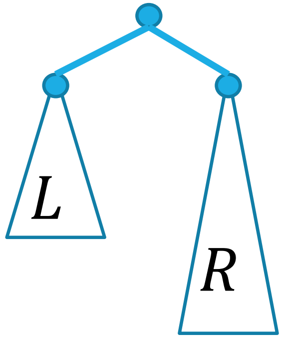
If L is taller than R?

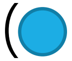


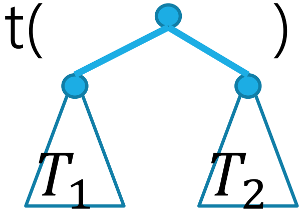
If L, R same height?



If R is taller than L?



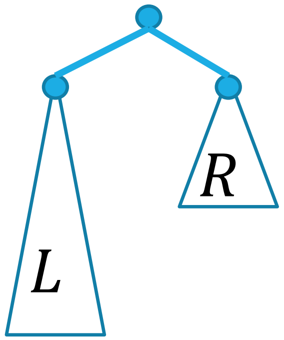
height() = 0

height() =

$1 + \max(\text{height}(T_1), \text{height}(T_2))$

How do heights compare?

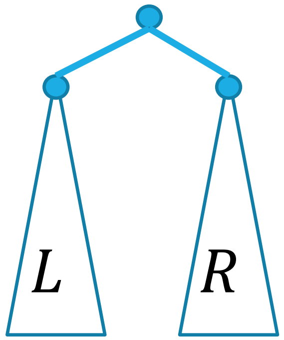
If L is taller than R?



height(T) = height$(L) + 1$

height(T) > height$(R) + 1$

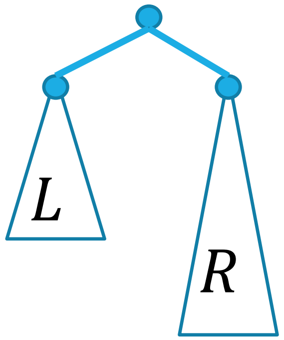
If L, R same height?



height(T) = height$(L) + 1$

height(T) = height$(R) + 1$

If R is taller than L?



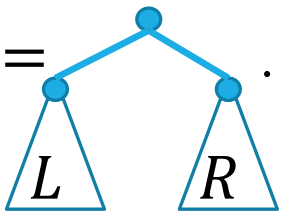
height(T) > height$(L) + 1$

height(T) = height$(R) + 1$

In all cases: height$(T) \geq$ height$(L) + 1$, height$(T) \geq$ height$(R) + 1$

Structural Induction on Binary Trees (cont.)

Let $P(T)$ be "size(T) $\leq 2^{height(T)+1} - 1$". We show $P(T)$ for all binary trees T by structural induction.

$T =$  .

$P(L)$: size$(L) \leq 2^{height(L)+1} - 1$

$P(R)$: size$(R) \leq 2^{height(R)+1} - 1$

height(T)=$1 + \max\{height(L), height(R)\}$

size(T)= 1 +size(L)+size(R)

size(T)=1+size(L)+size(R) $\leq 1 + 2^{height(L)+1} - 1 + 2^{height(R)+1} - 1$ (by IH)

$\leq 2^{height(L)+1} + 2^{height(R)+1} - 1$ (cancel 1's)

$\leq 2^{height(T)} + 2^{height(T)} - 1 = 2^{height(T)+1} - 1$ (T taller than subtrees)

So $P(T)$ holds, and we have $P(T)$ for all binary trees T by the principle of induction.

Structural Induction Template

1. Define $P()$ State that you will show $P(x)$ holds for all $x \in S$ and that your proof is by structural induction.
2. Base Case: Show $P(b)$
   [Do that for every b in the basis step of defining S]
3. Inductive Hypothesis: Suppose $P(x)$
   [Do that for every x listed as already in S in the recursive rules].
4. Inductive Step: Show $P()$ holds for the "new elements."
   [You will need a separate step for every element created by the recursive rules].
5. Therefore $P(x)$ holds for all $x \in S$ by the principle of induction.

Structural Induction on Strings

Strings

ε is "the empty string"

The string with 0 characters – `""` in Java (not `null!`)

Σ^*:

Basis: $\varepsilon \in \Sigma^*$.

Recursive: If $w \in \Sigma^*$ and $a \in \Sigma$ then $wa \in \Sigma^*$

wa means the string of w with the character a appended.

You'll also see $w \cdot a$ (a $\cdot$ to mean "concatenate" i.e. + in Java)

Functions on Strings

Since strings are defined recursively, most functions on strings are as well.

Length:

len(ε)=0;

len(wa)=len(w)+1 for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R = \varepsilon$;
$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation

$x \cdot \varepsilon = x$ for all $x \in \Sigma^*$;
$x \cdot (wa) = (x \cdot w)a$ for $w \in \Sigma^*, a \in \Sigma$

Number of c's in a string

$\#_c(\varepsilon) = 0$
$\#_c(wc) = \#_c(w) + 1$ for $w \in \Sigma^*$;
$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*, a \in \Sigma \setminus \{c\}$.

Claim for all $x, y \in \Sigma^*$ len(x·y)=len(x) + len(y).

Let $P(y)$ be "for all $x \in \Sigma^*$ len(x·y)=len(x) + len(y)."

Notice the strangeness of this $P()$ there is a "for all x" inside the definition of $P(y)$.

That means we'll have to introduce an arbitrary x as part of the base case and the inductive step!

Claim for all $x, y \in \Sigma^*$ len(x·y)=len(x) + len(y).

Let $P(y)$ be “len(x·y)=len(x) + len(y) for all $x \in \Sigma^*$.“

We prove $P(y)$ for all $x \in \Sigma^*$ by structural induction.

Base Case:

Inductive Hypothesis

Inductive Step:

We conclude that $P(y)$ holds for all string y by the principle of induction. Unwrapping the definition of P, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

Claim for all $x, y \in \Sigma^*$ len(x·y)=len(x) + len(y).

Let $P(y)$ be "len(x·y)=len(x) + len(y) for all $x \in \Sigma^*$."

We prove $P(y)$ for all $x \in \Sigma^*$ by structural induction.

Base Case: Let x be an arbitrary string, len($x \cdot \epsilon$)=len(x) =len(x)+0=len(x)+len(ε)

Inductive Hypothesis: Suppose $P(w)$ for an arbitrary string w.

Inductive Step:

We conclude that $P(y)$ holds for all string y by the principle of induction. Unwrapping the definition of P, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

Claim for all $x, y \in \Sigma^*$ len(x·y)=len(x) + len(y).

Let $P(y)$ be "len(x·y)=len(x) + len(y) for all $x \in \Sigma^*$. "

We prove $P(y)$ for all $x \in \Sigma^*$ by structural induction.

Base Case: Let x be an arbitrary string, len($x \cdot \epsilon$)=len(x) =len(x)+0=len(x)+len(ε)

Inductive Hypothesis: Suppose $P(w)$ for an arbitrary string w.

Inductive Step: Let $y = wa$ for an arbitrary $a \in \Sigma$. We show $P(y)$. Let x be an arbitrary string.

…

Therefore, len(xy)=len(x) + len(y), as required.

We conclude that $P(y)$ holds for all string y by the principle of induction. Unwrapping the definition of P, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

Claim for all $x, y \in \Sigma^*$ len(x·y)=len(x) + len(y).

Let $P(y)$ be "len(x·y)=len(x) + len(y) for all $x \in \Sigma^*$."

We prove $P(y)$ for all $x \in \Sigma^*$ by structural induction.

Base Case: Let x be an arbitrary string, len($x \cdot \epsilon$)=len(x) =len(x)+0=len(x)+len(ε)

Inductive Hypothesis: Suppose $P(w)$ for an arbitrary string w.

Inductive Step: Let $y = wa$ for an arbitrary $a \in \Sigma$. We show $P(y)$**.** Let x be an arbitrary string.

len(xy)=len(xwa) =len(xw)+1 (by definition of len)

=len(x) + len(w) + 1 (by IH)

=len(x) + len(wa) (by definition of len)

Therefore, len(xy)=len(x) + len(y), as required.

We conclude that $P(y)$ holds for all string y by the principle of induction. Unwrapping the definition of P, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

Why all those arbitraries?

Let $P(y)$ be "len(x·y)=len(x) + len(y) for all $x \in \Sigma^*$."

We prove $P(y)$ for all $x \in \Sigma^*$ by structural induction.

Base Case: Let x be an arbitrary string, len($x \cdot \epsilon$)=len(x) =len(x)+0=len(x)+len(ε)

Inductive Hypothesis: Suppose $P(w)$ for an arbitrary string w.

Inductive Step: Let $y = wa$ for an arbitrary $a \in \Sigma$. We show $P(y)$. Let x be an arbitrary string.

len(xy)=len(xwa) =len(xw)+1 (by definition of len)

=len(x) + len(w) + 1 (by IH)

=len(x) + len(wa) (by definition of len)

Therefore, len(xy)=len(x) + len(y), as required.

We conclude that $P(y)$ holds for all strings y by the principle of induction. Unwrapping the definition of P, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

$P(\varepsilon)$ is a for-all statement, introduce arbitrary variable to show for-all.

Needs to be arbitrary because it's in the IH (induction wouldn't show "all strings" otherwise)

Recursive rule says "every $a \in \Sigma$" so we need to argue for every a.

$P(y)$ is a for-all statement, introduce arbitrary variable to show for-all.

A few last comments

What does the inductive step look like?

Here’s a recursively-defined set:

Basis: $0 \in T$ and $5 \in T$

Recursive: If $x, y \in T$ then $x + y \in T$ and $x - y \in T$.

Let $P(x)$ be “$5|x$”

What does the inductive step look like?

Well there’s two recursive rules, so we have two things to show

Just the IS (you still need the other steps)

Let t be an arbitrary element of T not covered by the base case. By the exclusion rule $t = x + y$ or $t = x - y$ for $x, y \in T$.

Inductive hypothesis: Suppose $P(x)$ and $P(y)$ hold.

Case 1: $\mathrm{t} = x + y$

By IH $5|x$ and $5|y$ so $5a = x$ and $5b = y$ for integers a, b.

Adding, we get $x + y = 5a + 5b = 5(a + b)$. Since a, b are integers, so is $a + b$, and $P(x + y)$, i.e. $P(t)$, holds.

Case 2: $\mathrm{t} = x - y$

By IH $5|x$ and $5|y$ so $5a = x$ and $5b = y$ for integers a, b.

Subtracting, we get $x - y = 5a - 5b = 5(a - b)$. Since a, b are integers, so is $a - b$, and $P(x - y)$, i.e., $P(t)$, holds.

In all cases, we have $P(t)$. By the principle of induction, $P(x)$ holds for all $x \in T$.

If you don’t have a recursively-defined set

You won’t do structural induction.

You can do weak or strong induction though.

For example, Let $P(n)$ be “for all elements of S of “size” n <something> is true”

To prove “for all $x \in S$ of size n…” you need to start with “let x be an arbitrary element of size $k+1$ in your IS.

You CAN’T start with size k and “build up” to an arbitrary element of size $k+1$ it isn’t arbitrary.

Extra Practice

Induction: Hats!

You have n people in a line ($n \geq 2$). Each of them wears either a purple hat or a gold hat. The person at the front of the line wears a purple hat. The person at the back of the line wears a gold hat.

Show that for every arrangement of the line satisfying the rule above, there is a person with a purple hat next to someone with a gold hat.

Yes, this is kinda obvious. I promise this is good induction practice.

Yes, you could argue this by contradiction. I promise this is good induction practice.

Induction: Hats!

Define $P(n)$ to be "in every line of n people with gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show $P(n)$ for all integers $n \geq 2$ by induction on n.

Base Case: $n = 2$

Inductive Hypothesis:

Inductive Step:

By the principle of induction, we have $P(n)$ for all $n \geq 2$

Induction: Hats!

Define $P(n)$ to be "in every line of n people with gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show $P(n)$ for all integers $n \geq 2$ by induction on n.

Base Case: $n = 2$ The line must be just a person with a purple hat and a person with a gold hat, who are next to each other.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step: Consider an arbitrary line with $k + 1$ people in purple and gold hats, with a gold hat at one end and a purple hat at the other.

Target: there is someone in a purple hat next to someone in a gold hat.

By the principle of induction, we have $P(n)$ for all $n \geq 2$

Induction: Hats!

Define $P(n)$ to be "in every line of n people with gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show $P(n)$ for all integers $n \geq 2$ by induction on n.

Base Case: $n = 2$ The line must be just a person with a purple hat and a person with a gold hat, who are next to each other.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step: Consider an arbitrary line with $k + 1$ people in purple and gold hats, with a gold hat at one end and a purple hat at the other.

Case 1: There is someone with a purple hat next to the person in the gold hat at one end. Then those people are the required adjacent opposite hats.

Case 2:. There is a person with a gold hat next to the person in the gold hat at the end. Then the line from the second person to the end is length k, has a gold hat at one end and a purple hat at the other. Applying the inductive hypothesis, there is an adjacent, opposite-hat wearing pair.

In either case we have $P(k + 1)$.

By the principle of induction, we have $P(n)$ for all $n \geq 2$