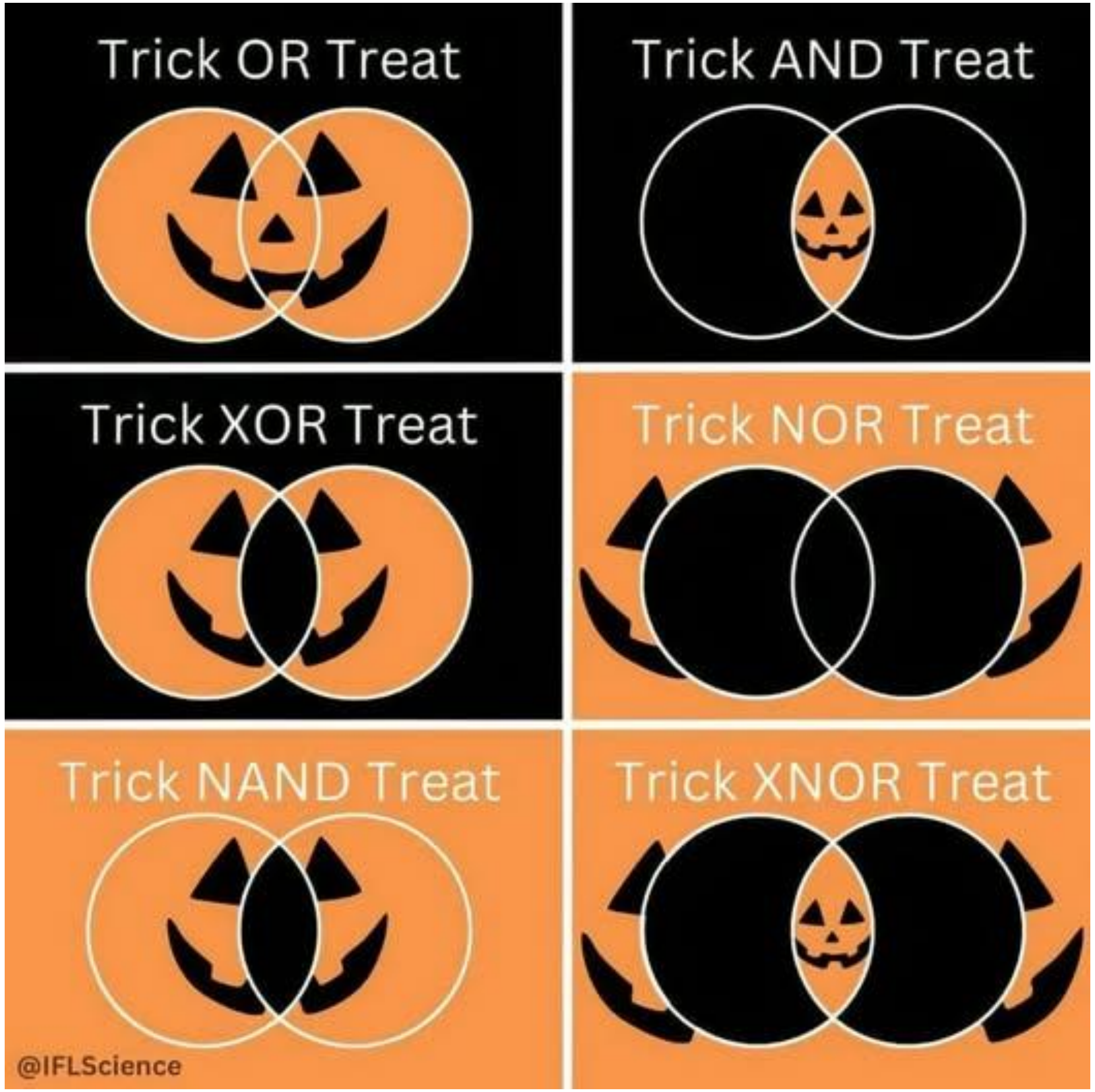


Sets

CSE 311 Autumn 25
Lecture 15

We've got A LOT of definitions today

You don't need me to read things aloud to you.

We'll cover the subtle/tricky things in lecture.

Other things are left for you to read on your own. The section is marked in the slide deck.

Today's concept check should be very useful to get the definitions down!

Set Basics 1

{1, 2, 3}   {3, 1, 2}

A set is an **unordered** group of **distinct** elements.

We'll always write a set as a list of its elements inside {curly, brackets}.

Variable names are capital letters, with lower-case letters for elements.

$|A| = 2$. "The size of A is 2." or "A has cardinality 2."

$A = \{\text{curly, brackets}\}$

$B = \{0,5,8,10\} = \{5,0,8,10\} = \{0,0,5,8,10\}$

$C = \{0,1,2,3,4, \dots\}$

Set Basics 2

$\in$ $\subseteq$

Some more symbols:

$a \in A$ ("a is in A" or "a is an element of A") means a is one of the members of the set.

For $B = \{0,5,8,10\}$, $0 \in B$.

$5 \in B$ $17 \notin B$

$A \subseteq B$ (A is a subset of B) means every element of A is also in B.

For $A = \{1,2\}$, $B = \{1,2,3\}$ $A \subseteq B$

Set Basics 3

Be careful about these two operations:

If $A = \{1,2,3,4,5\}$

$\{1\} \subseteq A$, but $\{1\} \notin A$

$1 \in A$

$\in$ asks: is this item in that box?

$\subseteq$ asks: is everything in this box also in that box?

$B = \{\{1\},\{2\},\{3\}\}$

$\{1\} \in B$

$\{1\} \notin B$

$\{\{1\}\} \subseteq B$

$C = \{\{1\}, 2, \{\{3\}\}\}$

Try it! (setup)

Let $A = \{1,2,3,4,5\}$

$B = \{1,2,5\}$

- Is $A \subseteq A$?
- Is $B \subseteq A$?
- Is $A \subseteq B$?
- Is $\{1\} \in A$?
- Is $1 \in A$?

Try it! (answers)

Let $A = \{1,2,3,4,5\}$

$B = \{1,2,5\}$

Is $A \subseteq A$? Yes!

Is $B \subseteq A$? Yes

Is $A \subseteq B$? No

Is $\{1\} \in A$? No

Is $1 \in A$? Yes

Definitions

$A \subseteq B$ ("A is a subset of B") iff every element of A is also in B.

$$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$$

$A = B$ ("A equals B") iff A and B have identical elements.

$$A = B \equiv \forall x(x \in A \leftrightarrow x \in B) \equiv A \subseteq B \wedge B \subseteq A$$

$$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

Proof Skeleton for $\subseteq$

How would we show $A \subseteq B$?

$$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$$

Let x be an arbitrary element of A

...

So x is also in B.

Since x was an arbitrary element of A, we have that $A \subseteq B$.

Proof Skeleton for $=$

That wasn't a "new" skeleton! It's exactly what we always do when we want to prove $\forall x(P(x) \rightarrow Q(x))$!

What about $A = B$?

$$A = B \equiv \forall x(x \in A \leftrightarrow x \in B) \equiv A \subseteq B \wedge B \subseteq A$$

Just do two subset proofs!

i.e. $\forall x(x \in A \rightarrow x \in B)$ and $\forall x(x \in B \rightarrow x \in A)$

What do we do with sets?

We combined propositions with $\vee, \wedge, \neg$.

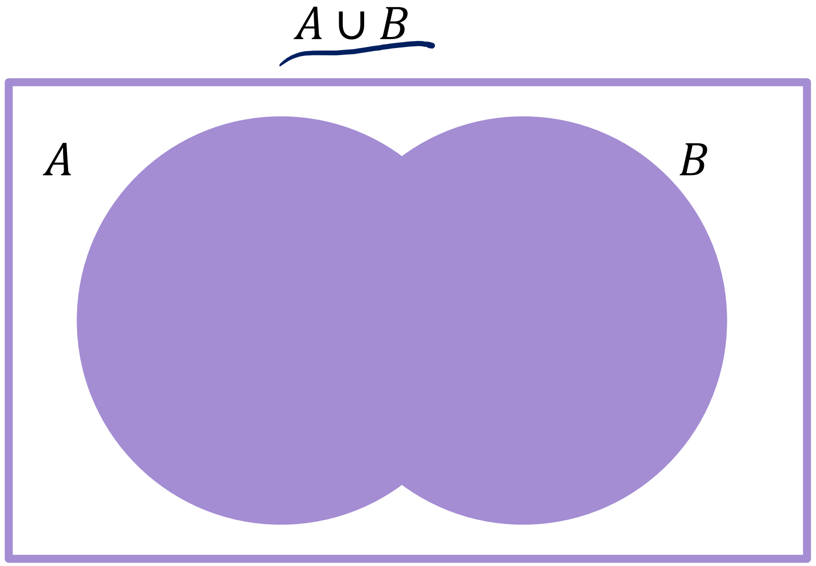
We combine sets with $\cap$ [intersection], $\cup$, [union] $\bar{}$ [complement]

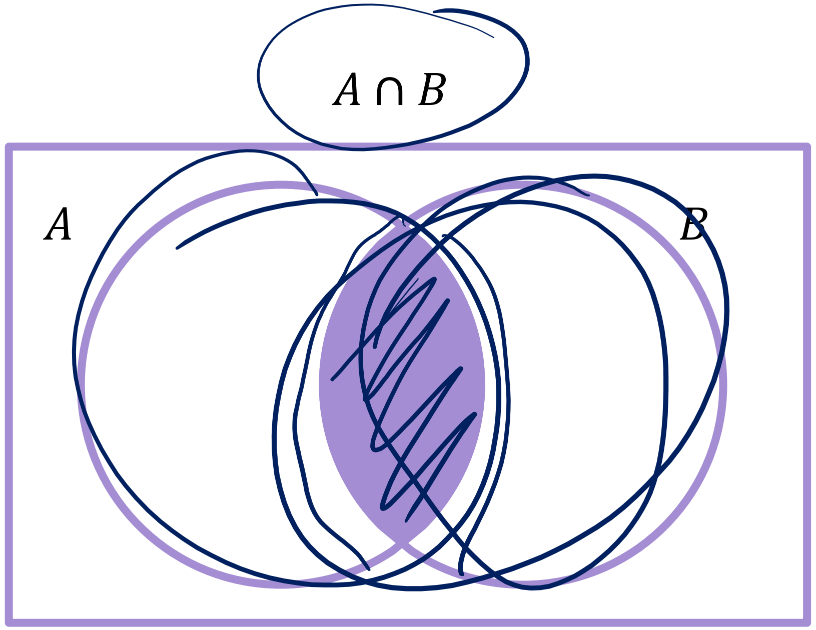
$A \cup B = \{x : x \in A \vee x \in B\}$

$A \cap B = \{x : x \in A \wedge x \in B\}$

$\bar{A} = \{x : x \notin A\}$

That's a lot of elements…if we take the complement, we'll have some "universe" $\mathcal{U}$, and $\bar{A} = \{x : x \in U \wedge x \notin A\}$
It's a lot like the domain of discourse.


A ∪ B
A
B


A ∩ B
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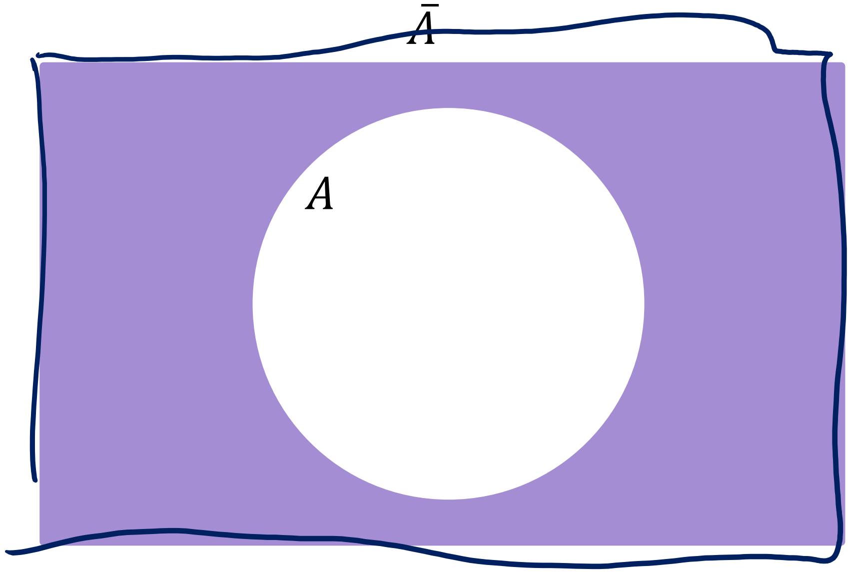

Ā
A

Proofs with sets

A proof!

What's the analogue of DeMorgan's Laws...

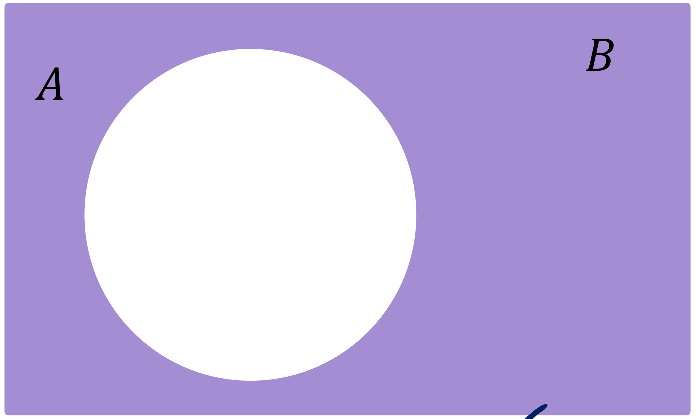
$\bar{A} \cap \bar{B} = \overline{A \cup B}$

$A = B \equiv \forall x(x \in A \leftrightarrow x \in B) \equiv A \subseteq B \land B \subseteq A$

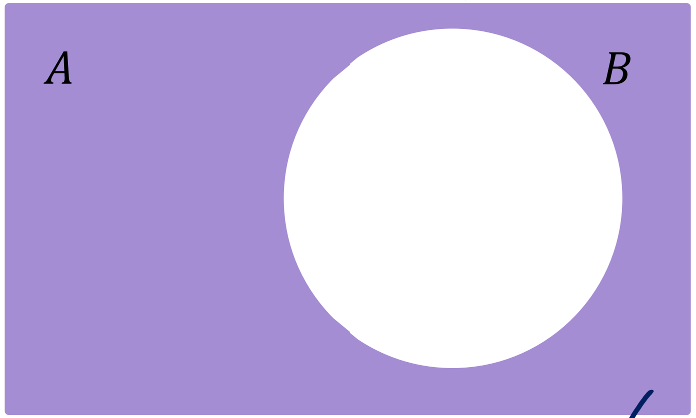
$\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$

$\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$

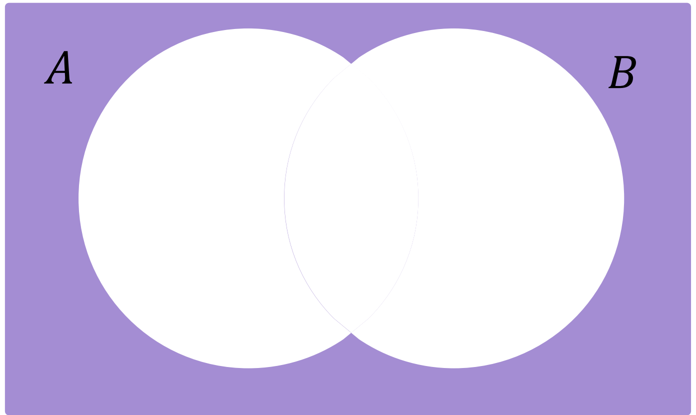
$\bar{A}$



$\bar{B}$



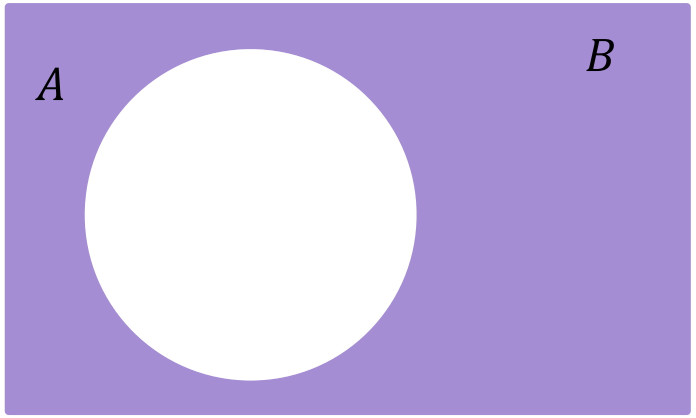
$\bar{A} \cap \bar{B}$



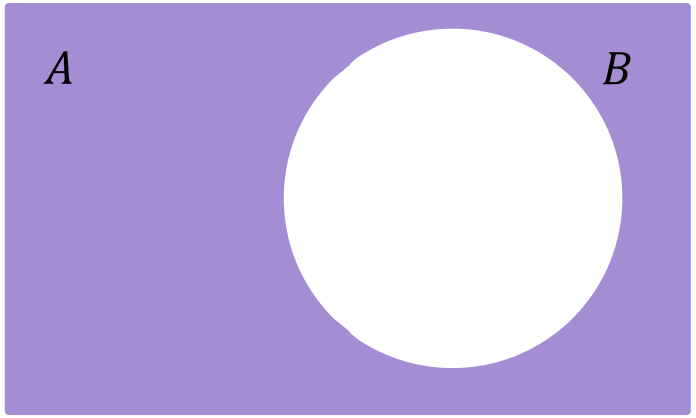
Try to find the diagram for $\overline{A \cup B}$

Is it the same?

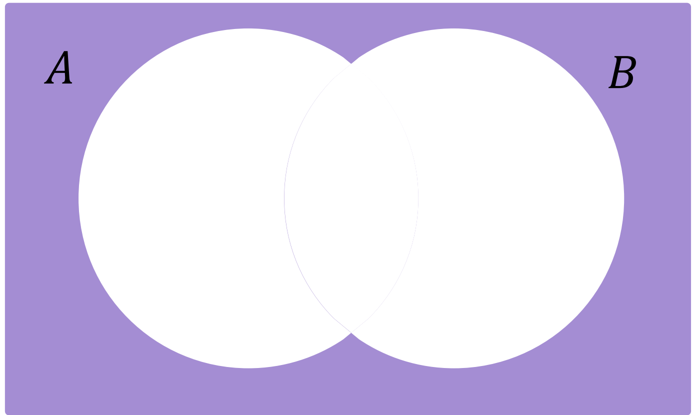
$\bar{A}$



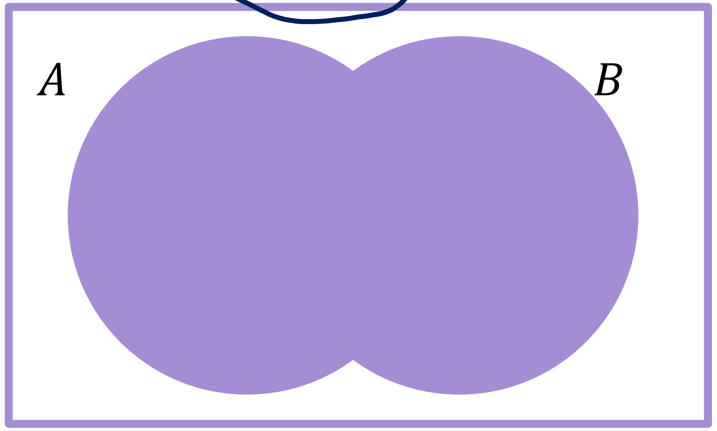
$\bar{B}$



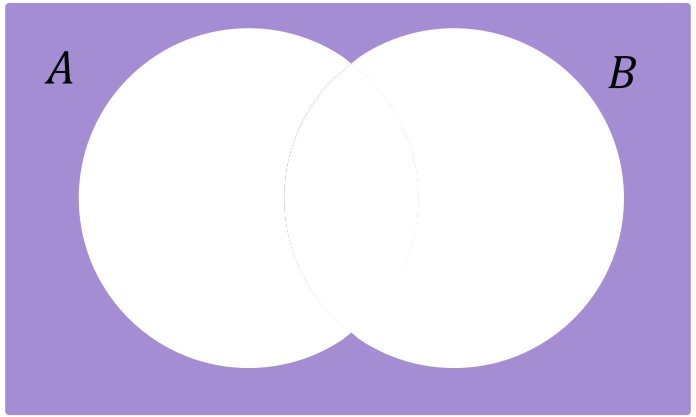
$\bar{A} \cap \bar{B}$



$A \cup B$



$\overline{A \cup B}$



A proof skeleton!

What's the analogue of DeMorgan's Laws…

$\bar{A} \cap \bar{B} = \overline{A \cup B}$

$A = B \equiv \forall x(x \in A \leftrightarrow x \in B) \equiv A \subseteq B \wedge B \subseteq A$

$\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$

Let x be an arbitrary element of $\bar{A} \cap \bar{B}$.

…

That is, x is in the complement of $A \cup B$, as required.

Since x was arbitrary $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$

$\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$

Let x be an arbitrary element of $\overline{A \cup B}$.

…

we get $x \in \bar{A} \cap \bar{B}$

Since x was arbitrary $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$

Since the subset relation holds in both directions, we have $\bar{A} \cap \bar{B} = \overline{A \cup B}$

$x \in \bar{A} \cap \bar{B}$

$x \in \bar{A}$ and $x \in \bar{B}$ not($x \in A \vee x \in B$)

$x \notin A \cup B$ not($x \in A \cup B$)

$x \in \overline{A \cup B}$

A half-complete proof!

What's the analogue of DeMorgan's Laws…

$\bar{A} \cap \bar{B} = \overline{A \cup B}$

$A = B \equiv \forall x(x \in A \leftrightarrow x \in B) \equiv A \subseteq B \land B \subseteq A$

$\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$

$\neg(x \in A) \land \neg(x \in B)$

Let x be an arbitrary element of $\bar{A} \cap \bar{B}$.

By definition of $\cap$ $x \in \bar{A}$ and $x \in \bar{B}$. By definition of complement, $x \notin A \land x \notin B$.

Applying DeMorgan's Law, we get $\neg(x \in A \lor x \in B)$.

Applying the definition of union, we get:$\neg(x \in A \cup B)$.

From the definition of complement, we get $x \in \overline{A \cup B}$, as required.

Since x was arbitrary $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$

…

Since the subset relation holds in both directions, we have $\bar{A} \cap \bar{B} = \overline{A \cup B}$

A complete proof!

What's the analogue of DeMorgan's Laws…

$\bar{A} \cap \bar{B} = \overline{A \cup B}$

$A = B \equiv \forall x(x \in A \leftrightarrow x \in B) \equiv A \subseteq B \land B \subseteq A$

$\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$
Let x be an arbitrary element of $\bar{A} \cap \bar{B}$.
By definition of $\cap$ $x \in \bar{A}$ and $x \in \bar{B}$. By definition of complement, $x \notin A \land x \notin B$.
Applying DeMorgan's Law, we get $\neg(x \in A \lor x \in B)$.
Applying the definition of union, we get:$\neg(x \in A \cup B)$.
From the definition of complement, we get $\mathrm{x} \in \overline{A \cup B}$, as required.
Since x was arbitrary $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$

$\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$
Let x be an arbitrary element of $\overline{A \cup B}$.
By definition of complement, x is not an element of $A \cup B$. Applying the definition of union, we get, $\neg(x \in A \lor x \in B)$
Applying DeMorgan's Law, we get: $x \notin A \land x \notin B$
By definition of complement, $x \in \bar{A} \land x \in \bar{B}$. So by definition of intersection, we get $x \in \bar{A} \cap \bar{B}$
Since x was arbitrary $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$

Since the subset relation holds in both directions, we have $\bar{A} \cap \bar{B} = \overline{A \cup B}$

Proof-writing advice

When you’re writing a set equality proof, often the two directions are nearly identical, just reversed.

It’s very tempting to use that $x \in A \leftrightarrow x \in B$ definition.

Be VERY VERY careful. It’s easy to mess that up, at every step you need to be saying “if and only if.”

Summary: How to show an if and only if

To show $p \leftrightarrow q$ you have two options:

Option A (STRONGLY recommended)

(1) $p \rightarrow q$

(2) $q \rightarrow p$

Option B (discouraged, but allowed)

p if-and-only-if p' if-and-only-if p'' if-and-only-if … if-and-only-if q

EVERY step must be an if-and-only if (in your justification AND explicitly written).

Two More Set Operations (Set-Builder)

In general
$\{\ variable : conditions\ \}$
Will also see | instead of :

Set-Builder Notation

Build your own set!

$\{x : \texttt{Conditions}(x)\}$

"The set of all x such that $\texttt{Conditions}(x)$"

Everything that meets the conditions (causes the expression after the : to be true) is in the set. Nothing else is.

$\{x : \texttt{Even}(x)\} = \{\dots, -4, -2, 0, 2, 4, \dots\}$

$\{y : \texttt{Prime}(y) \wedge \texttt{Even}(y)\} = \{2\}$

Build some sets! (setup)

Let $A = \{x : x\%3 = 0\}$

What are the elements of A?

Let $B = \{s : s \text{ is a string that contains at least one } 'a' \text{ and at least one } 'b'\}$

What are the elements of B?

Build some sets! (answers)

Let $A = \{x: x\%3 = 0\}$

What are the elements of A?

$\{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$

Let $B = \{s: s \text{ is a string that contains at least one 'a' and at least one 'b'}\}$

What are the elements of B?

$\{\ldots, \text{band, banana, abba, gumball}, ab, abab, bbaa, \ldots\}$

Two More Set Operations (Powerset)

Given a set, let's talk about it's powerset.

$\mathcal{P}(A) = \{X : X \text{ is a subset of } A\}$

The powerset of A is the **set** of all subsets of A.

$\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

$A = \{1, 2\}$

$\emptyset$

$|A| = 2$

$|\mathcal{P}(A)| = 4$

$\{\}$

$|A| = k$

$|\mathcal{P}(A)| = 2^k$

$\{1, 2, 3\}$

Recursive Set Definitions

Recursive Definition of Sets

Define a set S as follows:

Basis Step: $0 \in S$

Recursive Step: If $x \in S$ then $x + 2 \in S$.

Exclusion Rule: Every element of S is in S from the basis step (alone) or a finite number of recursive steps starting from a basis step.

What is S?

Recursive Definitions of Sets (laziness)

We'll always list the Basis and Recursive parts of the definition.

Starting…now…we're going to be lazy and skip writing the "exclusion" rule. It's still part of the definition.

Recursive Definitions of Sets (examples)

All Natural Numbers

Basis Step: $0 \in S$

Recursive Step: If $x \in S$ then $x + 1 \in S$.

All Integers

Basis Step: $0 \in S$

Recursive Step: If $x \in S$ then $x + 1 \in S$ and $x - 1 \in S$.

Integer coordinates in the line $y = x$

Basis Step: $(0,0) \in S$

Recursive Step: If $(x, y) \in S$ then $(x + 1, y + 1) \in S$ and $(x - 1, y - 1) \in S$.

Recursive Definitions of Sets (practice)

Q1: What is this set?

Basis Step: $6 \in S, 15 \in S$

Recursive Step: If $x, y \in S$ then $x + y \in S$

Q2: Write a recursive definition for the set of powers of 3 $\{1,3,9,27, \ldots\}$

Basis Step:

Recursive Step:

Extra Set Practice

Extra Set Practice (example 1, p1)

Show $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Proof:

Start with the outline. What **two** things do we need to show? For each, where do we start and end?

Extra Set Practice (example 1, p2)

Show $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Proof:
First, we'll show: $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Let x be an arbitrary element of$A \cup (B \cap C)$.
…
$x \in (A \cup B) \cap (A \cup C)$. So $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Now we show $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$
Let x be an arbitrary element of $(A \cup B) \cap (A \cup C)$.
…
$x \in A \cup (B \cap C)$. So $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Combining the two directions, since both sets are subsets of each other, we have:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Extra Set Practice (example 1, p3)

Show $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Proof:
First, we'll show: $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
Let x be an arbitrary element of$A \cup (B \cap C)$.
Then by definition of ∪,∩ we have:
$x \in A \vee (x \in B \wedge x \in C)$
Applying the distributive law, we get
$(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$
Applying the definition of union, we have:
$x \in (A \cup B)$ and $x \in (A \cup C)$
By definition of intersection we have $x \in (A \cup B) \cap (A \cup C)$.
So $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Now we show $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$
Let x be an arbitrary element of $(A \cup B) \cap (A \cup C)$.
By definition of intersection and union, $(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$
Applying the distributive law, we have $x \in A \vee (x \in B \wedge x \in C)$
Applying the definitions of union and intersection, we have $x \in A \cup (B \cap C)$
So $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.
Combining the two directions, since both sets are subsets of each other, we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Extra Set Practice (example 2, p1)

Suppose $A \subseteq B$. Show that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Extra Set Practice (example 2, p2)

Suppose $A \subseteq B$. Show that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Let A, B be arbitrary sets such that $A \subseteq B$.

Let X be an arbitrary element of $\mathcal{P}(A)$.

Thus $X \in \mathcal{P}(B)$ by definition of powerset.

Since an arbitrary element of $\mathcal{P}(A)$ is also in $\mathcal{P}(B)$, we have $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Extra Set Practice (example 2, p3)

Suppose $A \subseteq B$. Show that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Let A, B be arbitrary sets such that $A \subseteq B$.

Let X be an arbitrary element of $\mathcal{P}(A)$.

By definition of powerset, $X \subseteq A$.

Since $X \subseteq A$, every element of X is also in A. And since $A \subseteq B$, we also have that every element of X is also in B.

Thus $X \in \mathcal{P}(B)$ by definition of powerset.

Since an arbitrary element of $\mathcal{P}(A)$ is also in $\mathcal{P}(B)$, we have $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Extra Set Practice (example 3, p1)

Disprove: If $A \subseteq (B \cup C)$ then $A \subseteq B$ or $A \subseteq C$

Extra Set Practice (example 3, p2)

Disprove: If $A \subseteq (B \cup C)$ then $A \subseteq B$ or $A \subseteq C$

Consider $A = \{1,2,3\}, B = \{1,2\}, C = \{3,4\}$.

$B \cup C = \{1,2,3,4\}$ so we do have $A \subseteq (B \cup C)$, but $A \nsubseteq B$ and $A \nsubseteq C$.

When you disprove a ∀, you're just providing a counterexample (you're showing ∃) – your proof won't have "let x be an arbitrary element of A."

Read on Your Own

Some old friends (and some new ones)

$\mathbb{N}$ is the set of **Natural Numbers;** $\mathbb{N} = \{0, 1, 2, \ldots\}$

$\mathbb{Z}$ is the set of **Integers**; $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Q}$ is the set of **Rational Numbers**; e.g. ½, -17, 32/48

$\mathbb{R}$ is the set of **Real Numbers**; e.g. 1, -17, 32/48, $\pi, \sqrt{2}$

[n] is the set **{1, 2, ..., n}** when **n** is a positive integer

{} = ∅ is the **empty set**; the *only* set with no elements

Some old friends (and some new ones) with notes

Our natural numbers start at 0.
Common in CS, other resources start at 1.

$\mathbb{N}$ is the set of **Natural Numbers; $\mathbb{N}$** = {0, 1, 2, …}

$\mathbb{Z}$ is the set of **Integers**; $\mathbb{Z}$ = {…, -2, -1, 0, 1, 2, …}

$\mathbb{Q}$ is the set of **Rational Numbers**; e.g. ½, -17, 32/48

$\mathbb{R}$ is the set of **Real Numbers**; e.g. 1, -17, 32/48, π, $\sqrt{2}$

[n] is the set **{1, 2, …, n}** when **n** is a positive integer

{} = ∅ is the **empty set**; the *only* set with no elements

For the real numbers:
In LaTeX \mathbb{R}
In Office \doubleR

Use the empty set symbol ∅, not {}.
In LaTex \varnothing In Office \emptyset.

More Connectors! (1)

$A \setminus B$ "A minus B"

$$A \setminus B = \{x: x \in A \wedge x \notin B\}$$

$A \oplus B$ "XOR" (also called "symmetric difference")

$$A \oplus B = \{x: x \in A \oplus x \in B\}$$

More Connectors! (2)

$A \times B = \{(a, b): a \in A \wedge b \in B\}$

Called “the Cartesian product” of A and B.

$\mathbb{R} \times \mathbb{R}$ is the “real plane” ordered pairs of real numbers.

$\{1,2\} \times \{1,2,3\} = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$