Inference Proof

CSE 311 Fall 25
Lecture 8

Proof Strategy: Direct Proof

Our First Direct Proof (1)

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove: "For all integers x, if x is even, then x^2 is even." $\forall x \left(\text{Even}(x) \rightarrow \text{Even}\left(x^2\right)\right)$

Arbitrary

An "arbitrary" variable is one that

- is part of the domain of discourse (or some sub-domain you pick).
- You know **nothing** else about.

EVERY element of the domain could be plugged into that arbitrary variable. And everything else you say in the proof will follow.

An arbitrary variable is exactly what you need to convince us of a $\forall$.

If you want to prove a for-all you must explicitly tell us the variable is arbitrary when it is introduced.

- Your reader doesn't know what you're doing otherwise.

Our First Direct Proof (2)

<u>Definitions</u>
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove: "For all integers x, if x is even, then x^2 is even." $\forall x \left(\text{Even}(x) \rightarrow \text{Even}(x^2)\right)$

Proof: Let x be an arbitrary integer. Suppose that x is even.

Our First Direct Proof (Complete)

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove: "For all integers x, if x is even, then x^2 is even." $\forall x\left(\text{Even}(x) \to \text{Even}\left(x^2\right)\right)$

Proof: Let x be an arbitrary integer. Suppose that x is even.

By definition of even, $x = 2k$ for some integer k.

Squaring both sides, we see that:

$x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$

Because k is an integer, $2k^2$ is also an integer.

So x^2 is two times an integer.

Which is exactly the definition of even, so x^2 is even.

Since x was an arbitrary integer, we conclude that for all integers x, if x is even then x^2 is also even.

Direct Proof

Direct proof is one strategy for proving statements of the form

$$\forall x[P(x) \to Q(x)]$$

Direct Proof Template

Prove: $\forall x\left(\mathrm{Even}(x) \rightarrow \mathrm{Even}(x^2)\right)$

Declare an arbitrary variable for each ∀.	Let x be an arbitrary integer.
Assume the left side of the implication.	Suppose that x is even.
Unroll the predicate definitions.	Then by definition of even, there exists some integer k such that $x = 2k$.
Manipulate towards the goal.	Squaring both sides, we see that: $x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$
Reroll definitions into the right side of the implication.	Because k is an integer, then $2k^2$ is also an integer. So x^2 is two times an integer. So by definition of even, x^2 is even.
Conclude that you have proved the claim.	Since x was an arbitrary integer, we can conclude that for all integers x, if x is even then x^2 is even.

Direct Proof Steps

These are the usual steps. We'll see different outlines in the future!!

- Introduction
  - Declare an arbitrary variable for each ∀ quantifier
  - Assume the left side of the implication
- Core of the proof
  - Unroll the predicate definitions
  - Manipulate towards the goal (using creativity, algebra, etc.)
  - Reroll definitions into the right side of the implication
- Conclude that you have proved the claim

Inference Proofs

A Brief Return to Training Wheels (1)

For about 1.5 lectures, we're going to study "inference proofs"

The rules for these proofs are

1. Strict enough that computers can check them (there are languages designed to do that!)

2. More general than the simplification rules we've seen so far.

   You'll still use the simplification rules!

   But you'll find we can prove more things (at least without significant difficulty).

3. More similar to the proofs we spend most of the quarter writing.

A Brief Return to Training Wheels (2)

The claims and proofs are quite abstract!

Why spend time here?

Some computer scientists use the fully formal (computer-checkable) version of the rules.

- Our PL group here contains experts in these topics!

We want your takeaways to be

- In principle, any proof we write in this class could be made fully formal and checked.
- But it can be a lot of work, so we usually think and communicate in English. We’re people after all!

Inference Proofs Overview

A new way of thinking of proofs:

Here’s one way to get an iron-clad guarantee:

1. Write down all the facts we know.
2. Combine the things we know to derive new facts.
3. Continue until what we want to show is a fact.

Drawing Conclusions

You know “If it is raining, then I have my umbrella”

And “It is raining”

You should conclude…. I have my umbrella!

For whatever you conclude, convert the statement to propositional logic – will your statement hold for any propositions, or is it specific to raining and umbrellas?

I know $(p \rightarrow q)$ and p, I can conclude q
Or said another way: $[(p \rightarrow q) \land p] \rightarrow q$

Modus Ponens Idea

The inference from the last slide is always valid. I.e.

$$[(p \to q) \land p] \to q$$

Has only True rows in its truth table (it's a tautology)

Modus Ponens – a formal proof

$[(p \to q) \land p] \to q$	$\equiv [(\neg p \lor q) \land p] \to q$	Law of Implication
	$\equiv [p \land (\neg p \lor q)] \to q$	Commutativity
	$\equiv [(p \land \neg p) \lor (p \land q)] \to q$	Distributivity
	$\equiv [\mathrm{F} \lor (p \land q)] \to q$	Negation
	$\equiv [(p \land q) \lor \mathrm{F}] \to q$	Commutativity
	$\equiv [(p \land q)] \to q$	Identity
	$\equiv [\neg(p \land q)] \lor q$	Law of Implication
	$\equiv [\neg p \lor \neg q] \lor q$	DeMorgan's Law
	$\equiv \neg p \lor [\neg q \lor q]$	Associativity
	$\equiv \neg p \lor [q \lor \neg q]$	Commutativity
	$\equiv \neg p \lor \mathrm{T}$	Negation
	$\equiv \mathrm{T}$	Domination

Modus Ponens

The inference from the last slide is always valid. I.e.

$$[(p \to q) \wedge p] \to q \equiv \mathrm{T}$$

We use that inference A LOT

So often people gave it a name ("Modus Ponens")

So often…we don't have time to repeat that 12 line proof EVERY TIME.

Let's make this another law we can apply in a single step.

- Just like refactoring a method in code.

Notation – Laws of Inference

We're using the "$\rightarrow$ " symbol A LOT.

Too much

Some new notation to make our lives easier.

$$\frac{\text{If we know } \textbf{both } A \text{ and } B}{\therefore \quad \text{We can conclude any (or all) of } C, D} \qquad \frac{A, B}{\therefore \quad C, D}$$

"$\therefore$" means "therefore" – I knew A, B therefore I can conclude C, D.

$$\frac{p \rightarrow q, p}{\therefore \quad q}$$

Modus Ponens, i.e. $[(p \rightarrow q) \wedge p] \rightarrow q)$, in our new notation.

Another Proof

Let’s keep going.

I know “If it is raining then I have my umbrella” and “I do not have my umbrella”

I can conclude… It is not raining!

What’s the general form? $[(p \to q) \land \neg q] \to \neg p$

How do you think the proof will go?

If you had to convince a friend of this claim in English, how would you do it?

A proof!

We know $p \rightarrow q$ and $\neg q$; we want to conclude $\neg p$.
Let's try to prove it. Our goal is to list facts until our goal becomes a fact.
We'll number our facts, and put a justification for each new one.

A proof! (Complete)

We know $p \to q$ and $\neg q$; we want to conclude $\neg p$.
Let's try to prove it. Our goal is to list facts until our goal becomes a fact.
We'll number our facts, and put a justification for each new one.

1.	$p \to q$	Given
2.	$\neg q$	Given
3.	$\neg q \to \neg p$	Contrapositive of 1.
4.	$\neg p$	Modus Ponens on 3,2.

Try it yourselves

Suppose you know $p \to q, \neg s \to \neg q$,and p.
Give an argument to conclude s.

Try it yourselves (Complete)

Suppose you know $p \rightarrow q, \neg s \rightarrow \neg q$,and p.
Give an argument to conclude s.

1.	$p \rightarrow q$	Given
2.	$\neg s \rightarrow \neg q$	Given
3.	p	Given
4.	q	Modus Ponens 1,3
5.	$q \rightarrow s$	Contrapositive of 2
6.	s	Modus Ponens 5,4

That was abstract!

Imagine that instead someone had said:

If `next` is `null`, then we go down the else-branch

If the input list is non-empty, then we don't go down the else-branch.

This test uses a non-empty list as input.

Can you conclude anything?

So…why do the abstract proof?

Mostly to practice…

Though sometimes it’s helpful to make things abstract.

The more general you make a claim…

The more abstract it is, and therefore more difficult to understand on the surface…

But the more different contexts it can be used in.

More Inference Rules (Eliminate ∧)

We need a couple more inference rules.

These rules set us up to get facts in exactly the right form to apply the really useful rules.

A lot like commutativity and associativity in the propositional logic rules.

Eliminate ∧	$\dfrac{A \land B}{\therefore \quad A, B}$	$\dfrac{\text{I know the fact } A \land B}{\therefore \text{ I can conclude } A \text{ is a fact and } B \text{ is a fact } \textbf{separately}.}$

More Inference Rules

In total, we have two for ∧ and two for ∨, one to create the connector, and one to remove it.

Eliminate ∧

$$\frac{A \wedge B}{\therefore \quad A, B}$$

Intro ∧

$$\frac{A, B}{\therefore \quad A \wedge B}$$

Eliminate ∨

$$\frac{A \vee B, \neg A}{\therefore \quad B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

None of these rules are surprising, but they are useful.

Direct Proof Rule

The Direct Proof Rule

We've been implicitly using another "rule" in our English proofs, the direct proof rule

$$\frac{\text{Write a proof "given } A \text{ conclude } B\text{"}}{A \to B}$$

Direct Proof rule

$$\frac{A \Rightarrow B}{A \to B}$$

This rule is different from the others – $A \Rightarrow B$ is not a "single fact."
It's an observation that we've done a proof. (i.e. that we showed fact B starting from A.)

We will get a lot of mileage out of this rule.

Inference Rules

Eliminate ∧

$$\frac{A \wedge B}{\therefore \quad A, B}$$

Eliminate ∨

$$\frac{A \vee B, \neg A}{\therefore \quad B}$$

Intro ∧

$$\frac{A, B}{\therefore \quad A \wedge B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

Direct Proof rule

$$\frac{A \Rightarrow B}{A \rightarrow B}$$

Modus Ponens

$$\frac{P \rightarrow Q, P}{\therefore \quad Q}$$

You can still use all the propositional logic equivalences too!

How would you argue…

Let’s say you have a piece of code.

And you think **if** the code gets null input **then** a `NullPointerException` will be thrown.

How would you convince your friend?

You’d probably trace the code, assuming you would get null input.

The code was your **given**

The null input is an assumption

In general

How do you convince someone that $p \rightarrow q$ is true given some surrounding context/some surrounding givens?

You suppose p is true (you assume p)

And then you'll show q must also be true.

Just from p and the Given information.

Given: $((p \to q) \wedge (q \to r))$
Show: $(p \to r)$ (setup)

Here's an incorrect proof.

1.	$(p \to q) \wedge (q \to r)$	Given
2.	$p \to q$	Eliminate ∧ (1)
3.	$q \to r$	Eliminate ∧ (1)
4.	p	Given???
5.	q	Modus Ponens 4,2
6.	r	Modus Ponens 5,3
7.	$p \to r$	Direct Proof Rule

Given: $((p \to q) \land (q \to r))$
Show: $(p \to r)$

(with notes)

Here's an incorrect proof.

Proofs are supposed to be lists of facts.
Some of these "facts" aren't really facts...

1. $(p \to q) \land (q \to r)$
2. $p \to q$
3. $q \to r$ — Eliminate ∧ (1)
4. p — Given ????
5. q — Modus Ponens 4,2
6. r — Modus Ponens 5,3
7. $p \to r$ — Direct Proof Rule

These facts depend on p.
But p isn't known generally.
It was assumed for the purpose of proving $p \to r$.

Given: $((p \to q) \wedge (q \to r))$
Show: $(p \to r)$ (corrected)

Here's a corrected version of the proof.

1. $(p \to q) \wedge (q \to r)$ Given
2. $p \to q$ Eliminate ∧ 1
3. $q \to r$ Eliminate ∧ 1
    4.1 p Assumption
    4.2 q Modus Ponens 4.1,2
    4.3 r Modus Ponens 4.2,3
5. $p \to r$ Direct Proof Rule

When introducing an assumption to prove an implication: Indent, and change numbering.

When reached your conclusion, use the Direct Proof Rule to observe the implication is a fact.

The conclusion is an unconditional fact (doesn't depend on p) so it goes back up a level

Try it! (setup)

Given: $p \vee q$, $(r \wedge s) \rightarrow \neg q$, r.
Show: $s \rightarrow p$

Eliminate ∧

$$\frac{A \wedge B}{\therefore \quad A, B}$$

Eliminate ∨

$$\frac{A \vee B, \neg A}{\therefore \quad B}$$

Intro ∧

$$\frac{A; B}{\therefore \quad A \wedge B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

Direct Proof rule

$$\frac{A \Rightarrow B}{A \rightarrow B}$$

Modus Ponens

$$\frac{P \rightarrow Q; P}{\therefore \quad Q}$$

You can still use all the propositional logic equivalences too!

Try it! (Complete)

Given: $p \vee q$, $(r \wedge s) \rightarrow \neg q$, r.
Show: $s \rightarrow p$

1. $p \vee q$ Given
2. $(r \wedge s) \rightarrow \neg q$ Given
3. r Given
   - 4.1 s Assumption
   - 4.2 $r \wedge s$ Intro ∧ (3,4.1)
   - 4.3 $\neg q$ Modus Ponens (2, 4.2)
   - 4.4 $q \vee p$ Commutativity (1)
   - 4.5 p Eliminate ∨ (4.4, 4.3)
5. $s \rightarrow p$ Direct Proof Rule

Caution

Be careful! Logical inference rules can only be applied to **entire** facts. They cannot be applied to portions of a statement (our propositional equivalences could apply to subexpressions). Why not for inference rules?

Suppose we know $p \rightarrow q$, r. Can we conclude q?

1.	$p \rightarrow q$	Given
2.	r	Given
3.	$(p \vee r) \rightarrow q$	Introduce ∨ (1)
4.	$p \vee r$	Introduce ∨ (2)
5.	q	Modus Ponens 3,4.

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

Inference Proofs in Predicate Logic

Proofs with Quantifiers

We've done symbolic proofs with propositional logic.

To include predicate logic, we'll need some rules about how to use quantifiers.

Eliminate ∀	$\forall x\, P(x)$	$\therefore P(a)$ for any a
Intro ∃	$P(c)$ for some c	$\therefore \exists x\, P(x)$
Intro ∀	$P(a)$; a is arbitrary	$\therefore \forall x\, P(x)$
Eliminate ∃	$\exists x P(x)$	$\therefore P(c)$ for a fresh c

Let's see a good example, then come back to those "arbitrary" and "fresh" conditions.

Proof Using Quantifiers

Suppose we know $\exists x P(x)$ and $\forall y[\, P(y) \rightarrow Q(y)]$. Conclude $\exists x Q(x)$.

Eliminate ∀

$$\frac{\forall x\, P(x)}{\therefore P(a) \text{ for any } a}$$

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Intro ∀

$$\frac{P(a);\, a \text{ is arbitrary}}{\therefore \quad \forall x\, P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore \; P(c) \text{ for a fresh } c}$$

Example Proof Using Quantifiers (setup)

Suppose we know $\exists x P(x)$ and $\forall y[\, P(y) \rightarrow Q(y)]$. Conclude $\exists x Q(x)$.

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore \; P(c) \text{ for a fresh } c}$$

Eliminate ∀

$$\frac{\forall x\, P(x)}{\therefore P(a) \text{ for any } a}$$

Intro ∀

$$\frac{P(a); a \text{ is arbitrary}}{\therefore \quad \forall x\, P(x)}$$

Example Proof Using Quantifiers (Complete)

Suppose we know $\exists x P(x)$ and $\forall y[\, P(y) \rightarrow Q(y)]$. Conclude $\exists x Q(x)$.

1. $\exists x P(x)$ Given
2. $P(a)$ Eliminate ∃ 1
3. $\forall y[P(y) \rightarrow Q(y)]$ Given
4. $P(a) \rightarrow Q(a)$ Eliminate ∀ 3
5. $Q(a)$ Modus Ponens 2,4
6. $\exists x Q(x)$ Intro ∃ 5

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore \quad P(c) \text{ for a fresh } c}$$

Eliminate ∀

$$\frac{\forall x\, P(x)}{\therefore P(a) \text{ for any } a}$$

Intro ∀

$$\frac{P(a); a \text{ is arbitrary}}{\therefore \quad \forall x\, P(x)}$$

Proofs with Quantifiers (1)

We've done symbolic proofs with propositional logic.

To include predicate logic, we'll need some rules about how to use quantifiers.

Eliminate ∀

$$\frac{\forall x\, P(x)}{\therefore P(a) \text{ for any } a}$$

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \exists x\, P(x)}$$

Intro ∀

$$\frac{P(a); a \text{ is arbitrary}}{\therefore \forall x\, P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for a fresh } c}$$

"arbitrary" means a is "just" a variable in our domain.
It doesn't depend on any other variables and wasn't introduced with other information.

Proofs with Quantifiers (2)

We've done symbolic proofs with propositional logic.

To include predicate logic, we'll need some rules about how to use quantifiers.

Eliminate ∀

$$\frac{\forall x\, P(x)}{\therefore\ P(a) \text{ for any } a}$$

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore\ \exists x\, P(x)}$$

Intro ∀

$$\frac{P(a);\, a \text{ is arbitrary}}{\therefore\ \forall x\, P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore\ P(c) \text{ for a fresh } c}$$

> "fresh" means c is a new symbol (there isn't another c somewhere else in our proof).

Fresh and Arbitrary

Suppose we know $\exists x P(x)$. Can we conclude $\forall x P(x)$?

1. $\exists x\ P(x)$ Given
2. $P(a)$ Eliminate ∃ (1)
3. $\forall x\ P(x)$ Intro ∀ (2)

This proof is **definitely** wrong.
(take $P(x)$ to be "is a prime number")

a wasn't **arbitrary**. We knew something about it – it's the x that exists to make $P(x)$ true.

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\ P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore \quad P(c) \text{ for a fresh } c}$$

Eliminate ∀

$$\frac{\forall x\ P(x)}{\therefore\ P(a) \text{ for any } a}$$

Intro ∀

$$\frac{P(a);\ a \text{ is arbitrary}}{\therefore \quad \forall x\ P(x)}$$

Fresh and Arbitrary (notes 1)

Intro ∀

$$\frac{P(a);\ a \text{ is arbitrary}}{\therefore\quad \forall x\, P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore\quad P(c) \text{ for a fresh } c}$$

You can trust a variable to be **arbitrary** if you introduce it as such.

If you eliminated a ∀ to create a variable, that variable is arbitrary. Otherwise it's not arbitrary – it depends on something.

You can trust a variable to be **fresh** if the variable doesn't appear anywhere else (i.e. just use a new letter)

Fresh and Arbitrary (notes 2)

Eliminate ∀

$$\frac{\forall x\, P(x)}{\therefore P(a) \text{ for any } a}$$

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore\ \exists x\, P(x)}$$

There are no similar concerns with these two rules.

Want to reuse a variable when you eliminate ∀? Go ahead.

Have a c that depends on many other variables, and want to intro ∃?

Also not a problem.

Arbitrary Practice (1)

In section, you said: $[\exists y \forall x\, P(x, y)] \rightarrow [\forall x \exists y\, P(x, y)]$. Let's prove it!!

Arbitrary Practice (2)

In section, you said: $[\exists y \forall x\, P(x,y)] \rightarrow [\forall x \exists y\, P(x,y)]$. Let's prove it!!

1.1 $\exists y \forall x\, P(x,y)$	Assumption
1.2 $\forall x\, P(x,c)$	Elim $\exists$ (1.1)
1.3 Let a be arbitrary.	--
1.4 $P(a,c)$	Elim $\forall$ (1.2)
1.5 $\exists y\, P(a,y)$	Intro $\exists$ (1.4)
1.6 $\forall x \exists y\, P(x,y)$	Intro $\forall$ (1.5)
2. $[\exists y \forall x\, P(x,y)] \rightarrow [\forall x \exists y\, P(x,y)]$	Direct Proof Rule

Arbitrary Practice (3)

In section, you said: $[\exists y \forall x\, P(x,y)] \rightarrow [\forall x \exists y\, P(x,y)]$. Let's prove it!!

1.1 $\exists y \forall x\, P(x,y)$	Assumption
1.2 $\forall x\, P(x,c)$	Elim $\exists$ (1.1)
1.4 $P(a,c)$	Elim $\forall$ (1.2)
1.5 $\exists y\, P(a,y)$	Intro $\exists$ (1.4)
1.6 $\forall x \exists y\, P(x,y)$	Intro $\forall$ (1.5)

2. $[\exists y \forall x\, P(x,y)] \rightarrow [\forall x \exists y\, P(x,y)]$ Direct Proof Rule

It is not required to have "variable is arbitrary" as a step before using it. But many people (including Robbie) find it helpful.

Find The Bug

Let your domain of discourse be integers.
We claim that given $\forall x \exists y$ $\texttt{Greater}(y, x)$, we can conclude $\exists y \forall x$ $\texttt{Greater}(y, x)$
Where $\texttt{Greater}(y, x)$ means $y > x$

1.	$\forall x \exists y$ $\texttt{Greater}(y, x)$	Given
2.	Let a be an arbitrary integer	--
3.	$\exists y$ $\texttt{Greater}(y, a)$	Elim ∀ (1)
4.	$\texttt{Greater}(b, a)$	Elim ∃ (2)
5.	$\forall x$ $\texttt{Greater}(b, x)$	Intro ∀ (4)
6.	$\exists y \forall x$ $\texttt{Greater}(y, x)$	Intro ∃ (5)

Find The Bug (with notes)

1. $\forall x \exists y$ `Greater`(y, x) Given
2. Let a be an arbitrary integer --
3. $\exists y$ `Greater`(y, a) Elim ∀ (1)
4. `Greater`(b, a) Elim ∃ (2)
5. $\forall x$ `Greater`(b, x) Intro ∀ (4)
6. $\exists y \forall x$ `Greater`(y, x) Intro ∃ (5)

b is not a single number! The variable b depends on a. You can't get rid of a while b is still around.
What is b? It's probably something like $a + 1$.

Bug Found

There's one other "hidden" requirement to introduce $\forall$.

"No other variable in the statement can depend on the variable to be generalized"

Think of it like this -- b was probably $a + 1$ in that example.

You wouldn't have generalized from `Greater`$(a + 1, a)$

To $\forall x$ `Greater`$(a + 1, x)$. There's still an a, you'd have replaced all the a's.

x depends on y if y is in a statement when x is introduced.

This issue is much clearer in English proofs, which we'll start next time.

Extra Practice

One more Proof

Show if we know: $p, q, [(p \wedge q) \to (r \wedge s)], r \to t$ we can conclude t.

One more Proof (Complete)

Show if we know: $p, q, [(p \wedge q) \rightarrow (r \wedge s)], r \rightarrow t$ we can conclude t.

1.	p	Given
2.	q	Given
3.	$[(p \wedge q) \rightarrow (r \wedge s)]$	Given
4.	$r \rightarrow t$	Given
5.	$p \wedge q$	Intro ∧ (1,2)
6.	$r \wedge s$	Modus Ponens (3,5)
7.	r	Eliminate ∧ (6)
8.	t	Modus Ponens (4,7)