Our First Direct Proof (Complete)

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove: "For all integers x, if x is even, then x^2 is even." $\forall x \left(\text{Even}(x) \to \text{Even}(x^2)\right)$

Proof: Let x be an arbitrary integer. Suppose that x is even.

By definition of even, $x = 2k$ for some integer k.

Squaring both sides, we see that:

$x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$

Because k is an integer, $2k^2$ is also an integer.

So x^2 is two times an integer.

Which is exactly the definition of even, so x^2 is even.

Since x was an arbitrary integer, we conclude that for all integers x, if x is even then x^2 is also even.

Direct Proof Template

Prove: $\forall x \left(\text{Even}(x) \to \text{Even}(x^2)\right)$

Declare an arbitrary variable for each ∀.	Let x be an arbitrary integer.
Assume the left side of the implication.	Suppose that x is even.
Unroll the predicate definitions.	Then by definition of even, there exists some integer k such that $x = 2k$.
Manipulate towards the goal.	Squaring both sides, we see that: $x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$
Reroll definitions into the right side of the implication.	Because k is an integer, then $2k^2$ is also an integer. So x^2 is two times an integer. So by definition of even, x^2 is even.
Conclude that you have proved the claim.	Since x was an arbitrary integer, we can conclude that for all integers x, if x is even then x^2 is even.

Inference Rules

Eliminate ∧

$$\frac{A \wedge B}{\therefore \quad A, B}$$

Eliminate ∨

$$\frac{A \vee B, \neg A}{\therefore \quad B}$$

Intro ∧

$$\frac{A, B}{\therefore \quad A \wedge B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

Direct Proof rule

$$\frac{A \Rightarrow B}{A \rightarrow B}$$

Modus Ponens

$$\frac{P \rightarrow Q, P}{\therefore \quad Q}$$

You can still use all the propositional logic equivalences too!

Try it! (setup)

Given: $p \vee q$, $(r \wedge s) \rightarrow \neg q$, r.
Show: $s \rightarrow p$

Eliminate ∧ $\frac{A \wedge B}{\therefore \quad A, B}$

Eliminate ∨ $\frac{A \vee B, \neg A}{\therefore \quad B}$

Intro ∧ $\frac{A; B}{\therefore \quad A \wedge B}$

Intro ∨ $\frac{A}{\therefore A \vee B, B \vee A}$

Direct Proof rule $\frac{A \Rightarrow B}{A \rightarrow B}$

Modus Ponens $\frac{P \rightarrow Q; P}{\therefore \quad Q}$

You can still use all the propositional logic equivalences too!