Warm up translate to predicate logic:

"For every x, if x is prime, then x is odd or $x = 2$."

$\forall x(\text{Prime}(x) \rightarrow [\text{Odd}(x) \vee \text{Equal}(x,2)])$

Nested Unalike Quantifiers

CSE 311 Fall 25

Lecture 6

Announcements

HW1 solutions handed out in lecture today.

If you missed them, they are in an envelope outside Miya's office door

- CSE 460---in Allen take the elevator to the 4th floor, turn left.
- Don't need to wait for office hours, just grab them.
- Robbie will also have copies in office hours as well.

Where were we?

A predicate is a function that outputs a Boolean

`Prime(x)` := “`x` is prime”

`LessThan(x,y)`:= “`x<y`”

The “domain of discourse” is the set of all values your variables can take.

Usually the “type” you’re allowing

Quantifiers

We have two extra symbols to indicate which way we're using the variable.

1. The statement is true for every x, we just want to put a name on it.

$\forall x\ (\mathrm{p}(\mathrm{x}) \wedge q(x))$ means "for every x in our domain, $p(x)$ and $q(x)$ both evaluate to true."

2. There's some x out there that works, (but I might not know which it is, so I'm using a variable).

$\exists x(p(x) \wedge q(x))$ means "there is an x in our domain, such that $p(x)$ and $q(x)$ are both true.

Quantifiers

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Universal Quantifier

"$\forall x$"

"for each x", "for every x", "for all x" are common translations

Remember: upside-down-A for All.

Quantifiers

Existential Quantifier

"$\exists x$"

"there is an x", "there exists an x", "for some x" are common translations

Remember: backwards-E for Exists.

2. There's some x out there that works, (but I might not know which it is, so I'm using a variable).

$\exists x(p(x) \wedge q(x))$ means "there is an x in our domain, for which $p(x)$ and $q(x)$ are both true.

More Practice

Let your domain of discourse be fruits. Translate these

There is a fruit that is tasty and ripe.

For every fruit, if it is not ripe then it is not tasty.

There is a fruit that is sliced and diced.

More Practice

Let your domain of discourse be fruits. Translate these

There is a fruit that is tasty and ripe.

$$\exists x(\texttt{Tasty}(x) \land \texttt{Ripe}(x))$$

For every fruit, if it is not ripe then it is not tasty.

$$\forall x(\neg\texttt{Ripe}(x) \rightarrow \neg\texttt{Tasty}(x))$$

There is a fruit that is sliced and diced.

$$\exists x(\texttt{Sliced}(x) \land \texttt{Diced}(x))$$

Domain Restriction

Quantifiers

∀ (for **A**ll) and ∃ (there **E**xists)

Write these statements in predicate logic with quantifiers. Let your domain of discourse be "cats"

This sentence implicitly makes a statement about all cats!

If a cat is fat, then it is happy.

$$\forall x[\texttt{Fat}(x) \rightarrow \texttt{Happy}(x)]$$

Quantifiers

Writing implications can be tricky when we change the domain of discourse.

For every cat: if the cat is fat, then it is happy.

Domain of Discourse: cats $\forall x[\texttt{Fat}(x) \rightarrow \texttt{Happy}(x)]$

What if we change our domain of discourse to be all mammals?

We need to limit x to be a cat. How do we do that?

$\forall x[(\texttt{Cat}(x) \wedge \texttt{Fat}(x)) \rightarrow \texttt{Happy}(x)]$ $\forall x[\texttt{Cat}(x) \wedge (\texttt{Fat}(x) \rightarrow \texttt{Happy}(x))]$

Quantifiers

∀x [Cat(x) → (Fat(x) → Happy(x))]

Which of these translates "For every cat: if a cat is fat then it is happy."
when our domain of discourse is "mammals"?

$\forall x[(\texttt{Cat}(x) \wedge \texttt{Fat}(x)) \rightarrow \texttt{Happy}(x)]$

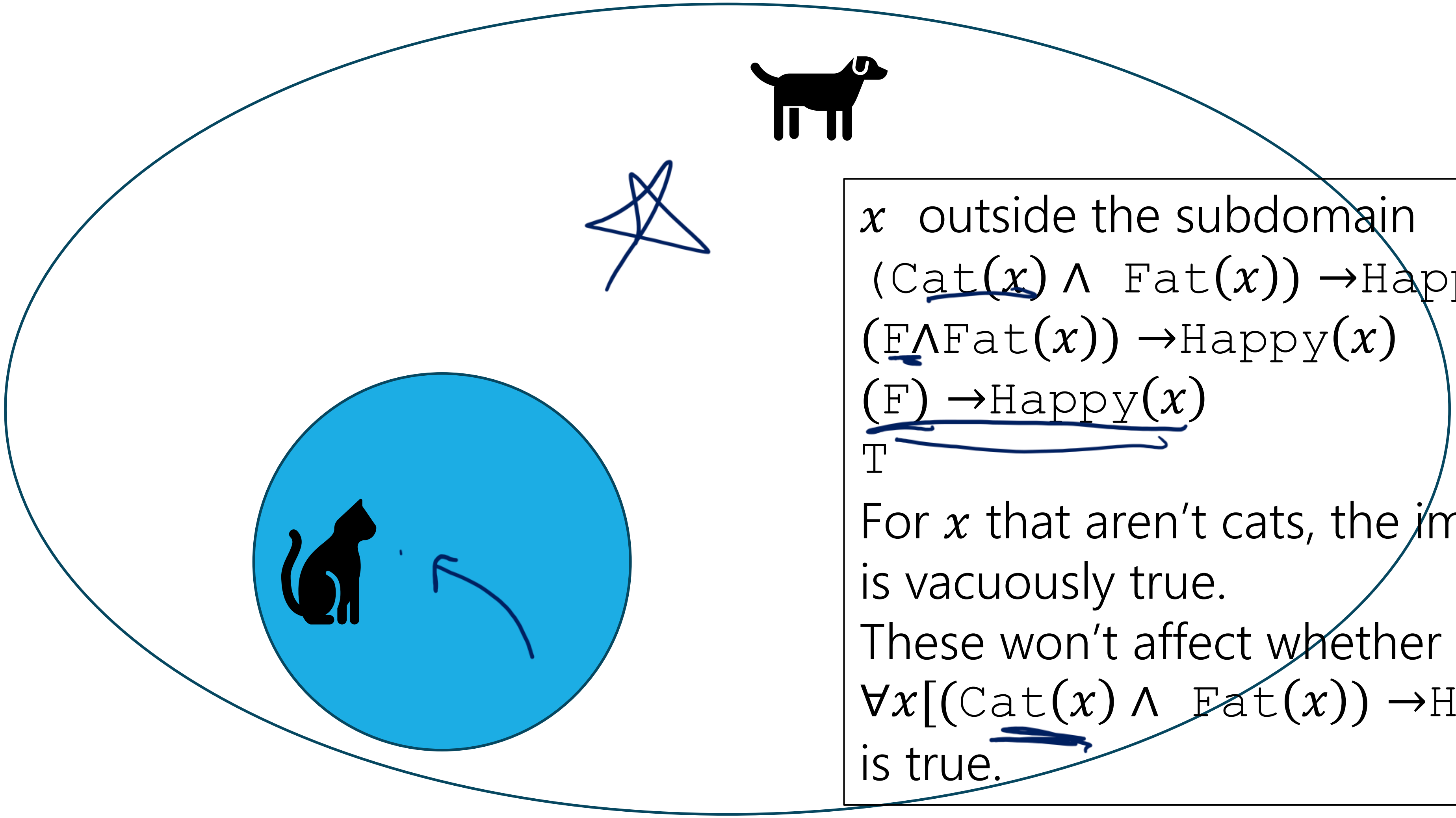
For all mammals, if x is a cat and fat then it is happy
[if x is not a cat, the claim is vacuously true, you can't use the promise for anything]

$\forall x[\texttt{Cat}(x) \wedge (\texttt{Fat}(x) \rightarrow \texttt{Happy}(x))]$

For all mammals, that mammal is a cat and if it is fat then it is happy.
[what if x is a dog? Dogs are in the domain, but...uh-oh. This isn't what we meant.]

To "limit" variables to a portion of your domain of discourse under a universal quantifier add a hypothesis to an implication.

$\forall x[(\mathtt{Cat}(x) \wedge \mathtt{Fat}(x)) \rightarrow \mathtt{Happy}(x)]$



x outside the subdomain

$(\mathtt{Cat}(x) \wedge \mathtt{Fat}(x)) \rightarrow \mathtt{Happy}(x)$

$(\mathtt{F} \wedge \mathtt{Fat}(x)) \rightarrow \mathtt{Happy}(x)$

$(\mathtt{F}) \rightarrow \mathtt{Happy}(x)$

$\mathtt{T}$

For x that aren't cats, the implication is vacuously true.

These won't affect whether $\forall x[(\mathtt{Cat}(x) \wedge \mathtt{Fat}(x)) \rightarrow \mathtt{Happy}(x)]$ is true.

$\forall x[(\mathrm{Cat}(x) \wedge \mathrm{Fat}(x)) \rightarrow \mathrm{Happy}(x)]$

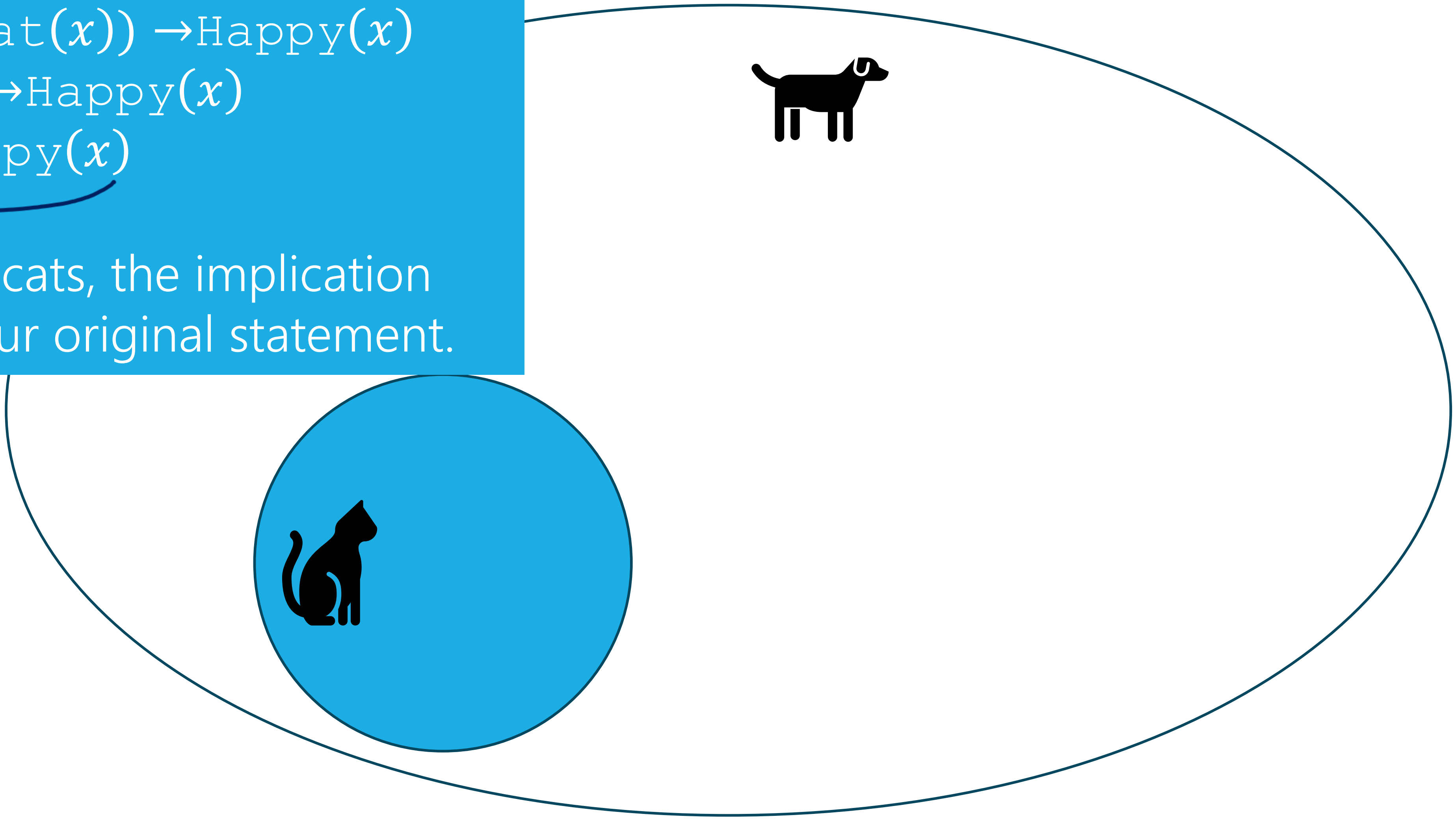
x inside the subdomain

$(\mathrm{Cat}(x) \wedge \mathrm{Fat}(x)) \rightarrow \mathrm{Happy}(x)$

$(\mathrm{T} \wedge \mathrm{Fat}(x)) \rightarrow \mathrm{Happy}(x)$

$\mathrm{Fat}(x) \rightarrow \mathrm{Happy}(x)$

For x that are cats, the implication simplifies to our original statement.



Quantifiers

Existential quantifiers need a different rule:

To "limit" variables to a portion of your domain of discourse under an existential quantifier AND the limitation together with the rest of the statement.

There is a dog who is not happy.

Domain of discourse: dogs

$\exists x(\neg \texttt{Happy}(x))$

Quantifiers

Which of these translates "There is a dog who is not happy."
when our domain of discourse is "mammals"?

$\exists x[\texttt{Dog}(x) \rightarrow \neg\texttt{Happy}(x)]$

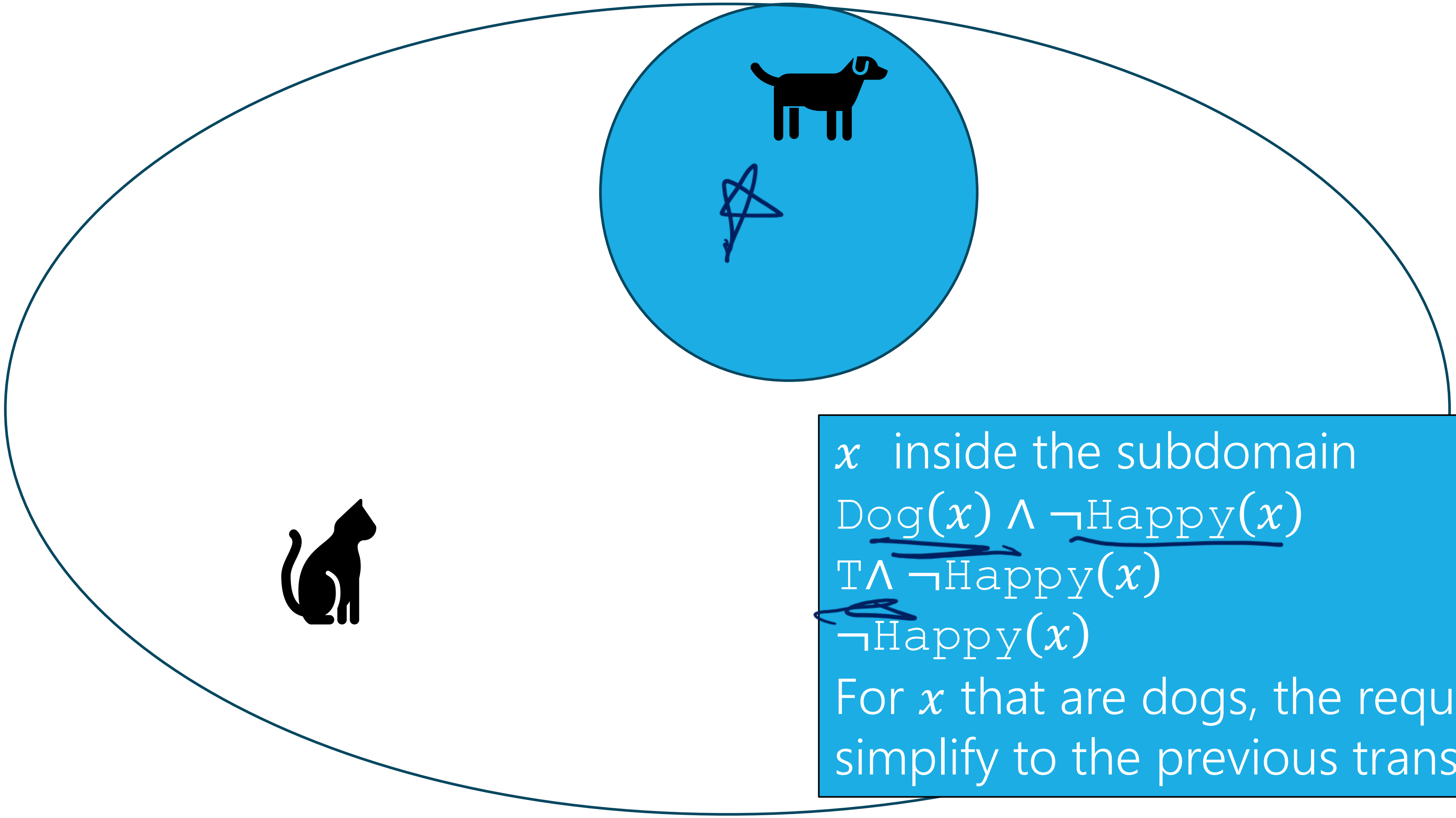
There is a mammal, such that if x is a dog then it is not happy.
[this can't be right – plug in a cat for x and the implication is true]

$\exists x[(\texttt{Dog}(x) \wedge \neg\texttt{Happy}(x))]$

There is a mammal that is both a dog and not happy.
[this one is correct!]

To "limit" variables to a portion of your domain of discourse under an existential quantifier AND the limitation together with the rest of the statement.

$\exists x[(\mathrm{Dog}(x) \land \neg\mathrm{Happy}(x))]$



x inside the subdomain

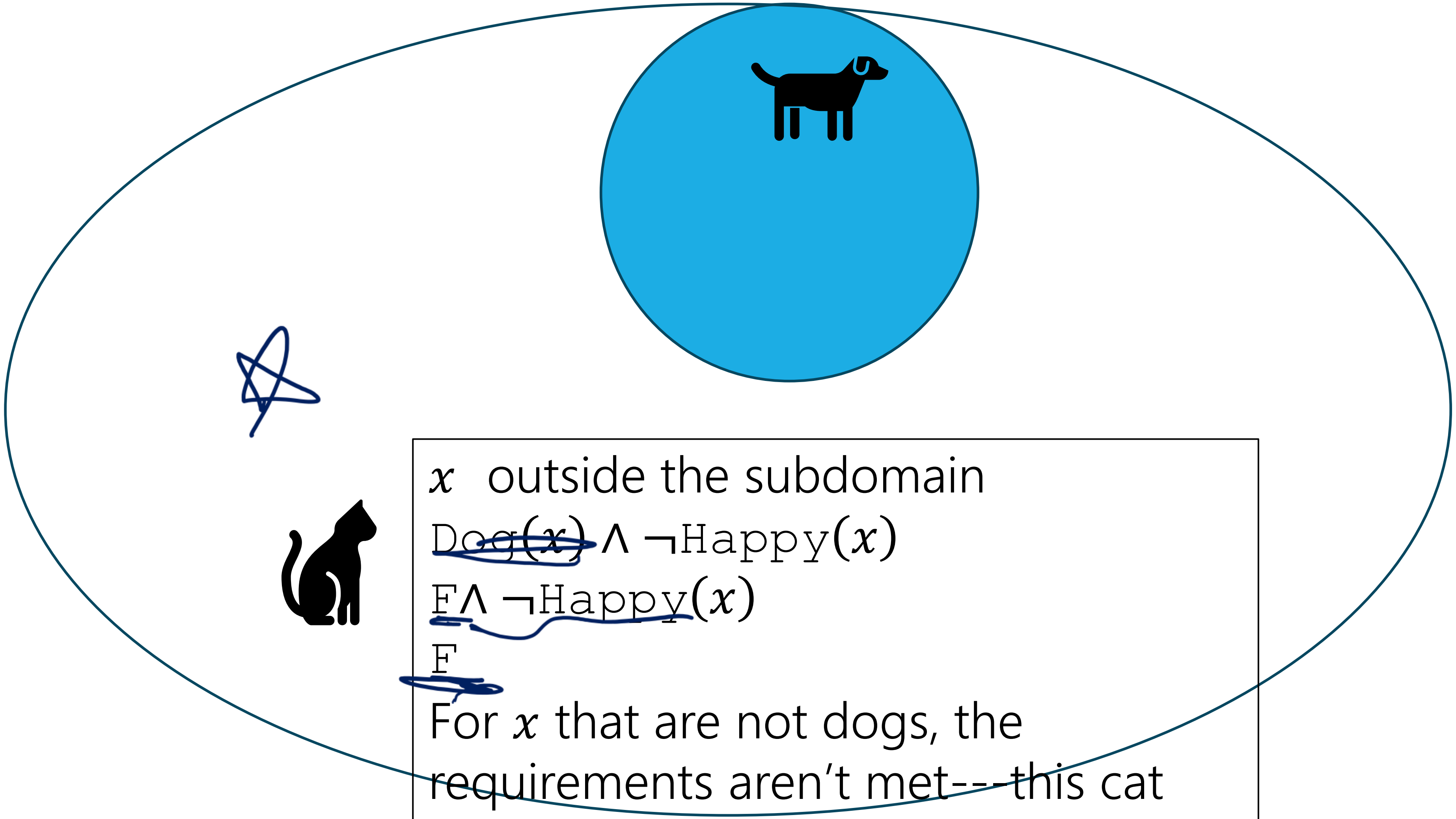
$\mathrm{Dog}(x) \land \neg\mathrm{Happy}(x)$

$\mathrm{T} \land \neg\mathrm{Happy}(x)$

$\neg\mathrm{Happy}(x)$

For x that are dogs, the requirements simplify to the previous translation

$\exists x[(\mathrm{Dog}(x) \land \neg\mathrm{Happy}(x))]$



x outside the subdomain

$\mathrm{Dog}(x) \land \neg\mathrm{Happy}(x)$

$\mathrm{F} \land \neg\mathrm{Happy}(x)$

F

For x that are not dogs, the requirements aren't met---this cat isn't what we're looking for.

Why are the rules what they are?

A universal quantifier is a "Big AND"

For a domain of discourse of $\{e_1, e_2, \dots, e_k\}$

$\forall x(P(x))$ means $P(e_1) \wedge P(e_2) \wedge \cdots \wedge P(e_k)$

Now let's say our domain is $\{e_1, e_2, \dots, e_k, f_1, f_2, \dots, f_j\}$ where f_i are the irrelevant parts of the bigger domain (non-cat-mammals). We want the expression to be

$P(e_1) \wedge P(e_2) \wedge \cdots \wedge P(e_k) \wedge T \wedge T \dots \wedge T$

$\forall x(RightSubDomain(x) \rightarrow P(x))$ does that!

Why are the rules what they are?

An existential quantifier is a "Big OR"

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$P(e_1) \vee P(e_2) \vee \cdots \vee P(e_k) \vee F \vee F \dots \vee F$

$\exists x(RightSubDomain(x) \wedge P(x))$ does that!

Negation

DoD: mammals

Translate these sentences to predicate logic, then negate them.

All cats have nine lives.

$\forall x\,(NumLives(x, 9))$

All dogs love every person.

There is a cat that loves someone.

Negation

$p \to q$

$\neg(\neg p \vee q) \equiv p \wedge \neg q$

Translate these sentences to predicate logic, then negate them.

All cats have nine lives.

$$\forall x\big(Cat(x) \to NumLives(x,9)\big)$$

$\exists x(Cat(x) \wedge \neg\big(NumLives(x,9)\big))$ “There is a cat without 9 lives.

All dogs love every person.

$$\forall x \forall y\big(Dog(x) \wedge Human(y) \to Love(x,y)\big)$$

$\exists x \exists y(Dog(x) \wedge Human(y) \wedge \neg Love(x,y))$ “There is a dog who does not love someone.” “There is a dog and a person such that the dog doesn’t love that person.”

There is a cat that loves someone.

$$\exists x \exists y(Cat(x) \wedge Human(y) \wedge Love(x,y)$$

$$\forall x \forall y(Cat(x) \wedge Human(y) \to \neg Love(x,y))$$

“For every cat and every human, the cat does not love that human.”

“Every cat does not love any human” (“no cat loves any human”)

Negation with Domain Restriction

$\exists x \exists y(Cat(x) \wedge Human(y) \wedge Love(x, y))$

$\forall x \forall y([Cat(x) \wedge Human(y)] \rightarrow \neg Love(x, y))$

There are lots of equivalent expressions to the second. This one is by far the best because it reflects the domain restriction happening. How did we get there?

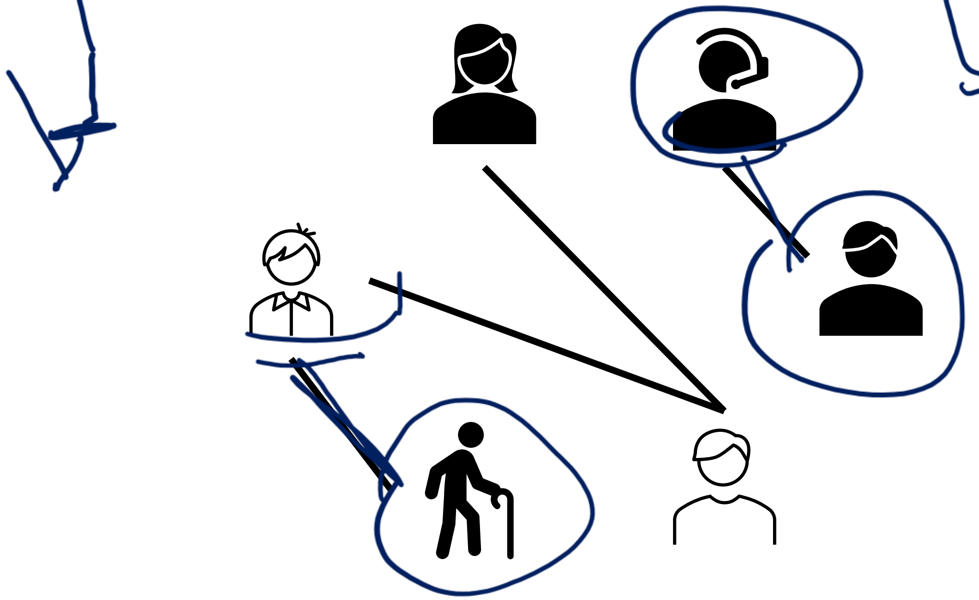
There's a problem in this week's section handout showing similar algebra.

Nested Quantifiers

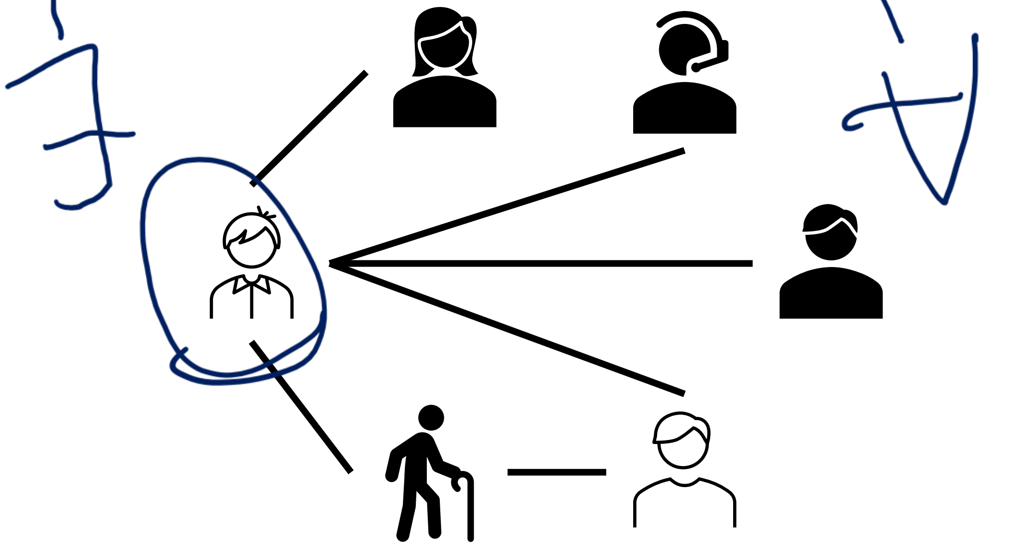
Nested Quantifiers

Translate these sentences using only quantifiers and the predicate `AreFriends`(x, y)

Everyone is friends with someone.



Someone is friends with everyone.



Nested Quantifiers

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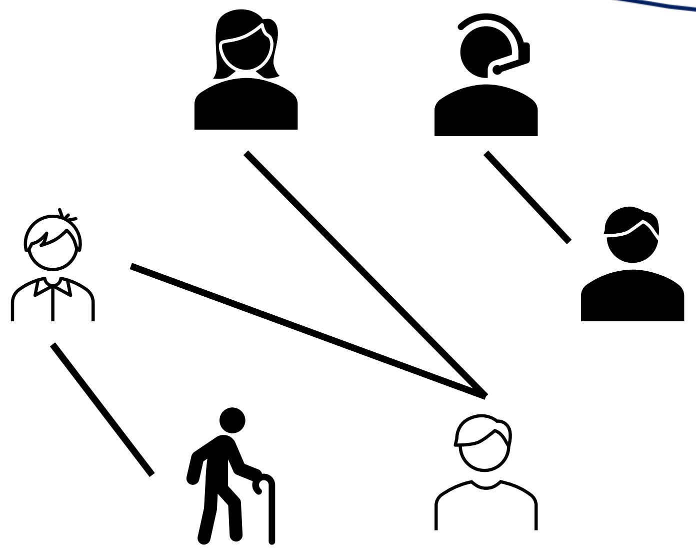

$\forall x(\exists y$ `AreFriends`$(x, y))$

$\forall x \exists y$ `AreFriends`(x, y)

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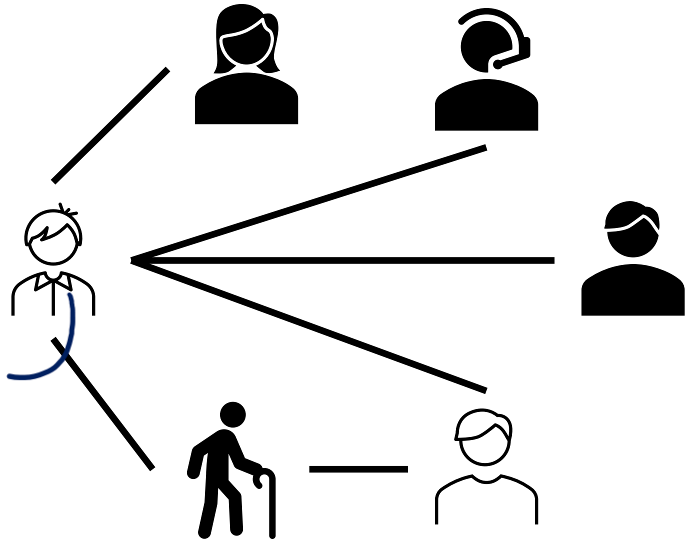

$\exists x(\forall y$ `AreFriends`$(x, y))$

$\exists x \forall y$ `AreFriends`(x, y)

Nested Quantifiers

$\forall x \exists y \, P(x, y)$

"For every x there exists a y such that $P(x, y)$ is true."

y might change depending on the x (people have different friends!).

$\exists x \forall y \, P(x, y)$

"There is an x such that for all y, $P(x, y)$ is true."

There's a special, magical x value so that $P(x, y)$ is true regardless of y.

Nested Quantifiers

Let our domain of discourse be $\{A, B, C, D, E\}$

And our proposition $P(x, y)$ be given by the table.

What should we look for in the table?

$\exists x \forall y P(x, y)$

$\forall x \exists y P(x, y)$

y (columns), x (rows)

$P(x, y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Nested Quantifiers

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And our proposition $P(x, y)$ be given by the table.

What should we look for in the table?

$\exists x \forall y P(x, y)$

A row, where every entry is T

$\forall x \exists y P(x, y)$

In every row there must be a T

$P(x,y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Keep everything in order

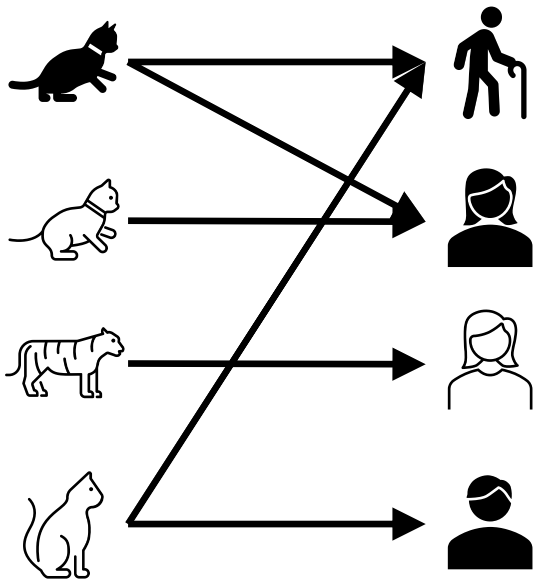
Keep the quantifiers in the same order in English as they are in the logical notation.

“There is someone out there for everyone” is a $\forall x \exists y$ statement in “everyday” English.

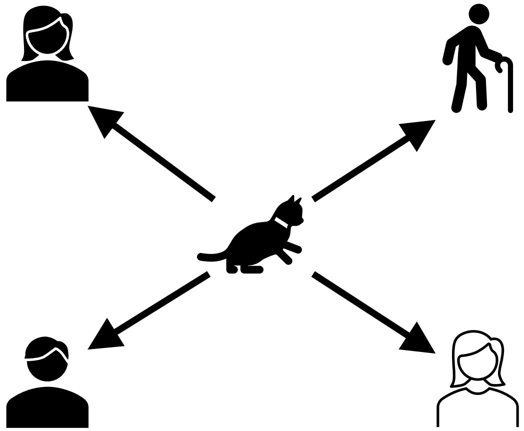
It would **never** be phrased that way in “mathematical English” We’ll only every write “for every person, there is someone out there for them.”

Try it yourselves

Every cat loves some human.



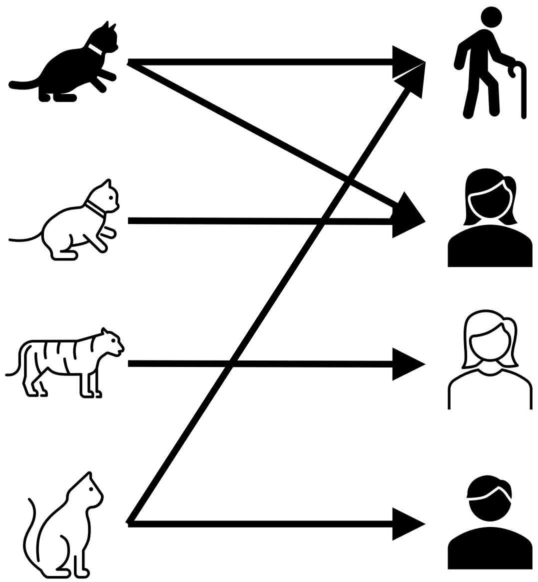
There is a cat that loves every human.



Let your domain of discourse be mammals.
Use the predicates $\texttt{Cat}(x)$, $\texttt{Dog}(x)$, and $\texttt{Loves}(x, y)$ to mean x loves y.

Try it yourselves

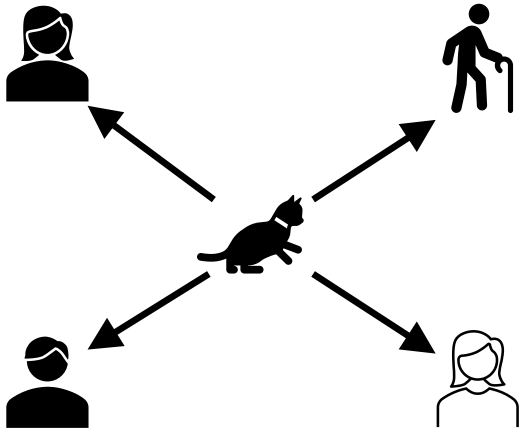
Every cat loves some human.



$\forall x\, (\mathtt{Cat}(x) \rightarrow \exists y[\mathtt{Human}(y) \land \mathtt{Loves}(x,y)])$

$\forall x \exists y(\mathtt{Cat}(x) \rightarrow [\mathtt{Human}(y) \land \mathtt{Loves}(x,y)])$

There is a cat that loves every human.



$\exists x\, (\mathtt{Cat}(x) \land \forall y[\mathtt{Human}(y) \rightarrow \mathtt{Loves}(x,y)])$

$\exists x \forall y(\mathtt{Cat}(x) \land [\mathtt{Human}(y) \rightarrow \mathtt{Loves}(x,y)])$

Negation

How do we negate nested quantifiers?

The old rule still applies.

To negate an expression with a quantifier
1. Switch the quantifier (∀ becomes ∃, ∃ becomes ∀)
2. Negate the expression inside

$$\neg(\forall x \exists y \forall z\, [P(x,y) \wedge Q(y,z)])$$

$$\exists x(\neg(\exists y \forall z\, [P(x,y) \wedge Q(y,z)]))$$

$$\exists x \forall y(\neg(\forall z[P(x,y) \wedge Q(y,z)]))$$

$$\exists x \forall y \exists z(\neg[P(x,y) \wedge Q(y,z)])$$

$$\exists x \forall y \exists z[\neg P(x,y) \vee \neg Q(y,z)]$$

More Translation

For each of the following, translate it, then say whether the statement is true. Let your domain of discourse be integers.

For every integer, there is a greater integer.

$\forall x \exists y(\texttt{Greater}(y, x))$ (This statement is true: y can be $x + 1$ [y depends on x])

There is an integer x, such that for all integers y, xy is equal to 1.

$\exists x \forall y(\texttt{Equal}(xy, 1))$ (This statement is false: no single value of x can play that role for every y.)

$\forall y \exists x(\texttt{Equal}(x + y, 1))$

For every integer, y, there is an integer x such that $x + y = 1$
(This statement is true, y can depend on x)