CSE 311 : Practice Midterm 2 Exam

1. Proof by Contrapositive

(a) Prove the following claim using proof by contrapositive

For all primes p, if p is greater than 2, then $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$.

Hint 1: You may use without proof that all prime numbers greater than 2 are odd.
Hint 2: You may use the fact that every integer p must be congruent to 0, 1, 2, or 3 mod 4.
Hint 3: If your contrapositive assumption leads to $p = 4k$ or $p = 4k + 2$, remember that for p to be prime, its only *positive* divisors must be 1 and p. Showing p has another divisor (like 2 or 4) means it cannot be prime (unless p itself is 2).

2. Set Theory

(a) Prove the following claim:

For all sets S and T, $\mathcal{P}(S) \cup \mathcal{P}(T) \subseteq \mathcal{P}(S \cup T)$.

Your proof must be in **English**. Do not write a logical equivalences proof. You can still use symbols within your **English proof** where appropriate.
Hint: Consider how the definition of union naturally creates cases (you can individually consider elements of the sets being unioned together). Then, see how the definition of powersets can be used to show elements of those sets are in $S \cup T$ and consider what this means for the powerset of $S \cup T$.

3. Induction

(a) Consider the following code snippet.

```
int Mystery(int n) {
    if n == 0:
        return 5
    if n == 1:
        return 16
    return 7 * Mystery(n - 1) - 10 * Mystery(n - 2)
}
```

In this problem, we will use `Mystery(n)` to refer to the value returned by the code snippet above when run on input `n`. For example, `Mystery(2)` $= (7 \cdot 16) - (10 \cdot 5) = 62$.

Use induction to show that `Mystery(n)` $= 3 \cdot 2^n + 2 \cdot 5^n$ for all integers $n \geq 0$. [20 points]

4. Proof by Contradiction

(a) Kuromi is buying boba for n friends. She notices that the total number of tapioca pearls, x, and the total price in cents, y, are related by the equation $x^2 - 2y^2 = 5$.

Prove by contradiction that at least one of the values, x or y, must be odd.

You may use the fact that an integer is even if and only if its square is even, and odd if and only if its square is odd.

5. Multiple Choice Questions

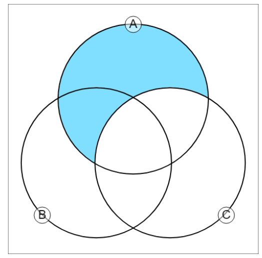
(a) Which of the following expressions represent the shaded area in the image? In the image, the top circle is A, the left circle is B, the right circle is C. **Select all that apply.** [3 points]

☐ $A \cap \overline{C}$

☐ $((A \cup B) \setminus C) \cap B$

☐ $\overline{A} \cap C$

☐ $((A \cup B) \setminus C) \cap A$



(b) How many elements are in $(\{1,2,3\} \cup \{2,4,6\}) \setminus \{5,6,7\}$

◯ 0
◯ 3
◯ 4
◯ 9

(c) Which of the following sets is the ordered pair $(1,2)$ an element of? **Select all that apply**.

☐ $\mathbb{Z} \times \mathbb{R}$
☐ $\mathcal{P}(\{1,2,3\})$
☐ $\{(x,y) : x, y \in \mathbb{Z}, y > x\}$
☐ $\{1,2,3,4,5\}$

(d) Which of the following statements are true? **Select all that apply.**

Let $A = \{x \in \mathbb{Z} : x \equiv 1 \pmod 3\}$ Let $B = \{x \in \mathbb{Z} : x \equiv 1 \pmod 6\}$ Let $C = \{x \in \mathbb{Z} : x = 6k + 1$ for some $k \in \mathbb{Z}\}$ Let $D = \{1, 7, 13\}$

☐ $B \subseteq A$
☐ $A \subseteq C$
☐ $B \subseteq C$
☐ $D \subseteq B$

(e) Consider the set S defined as:

- Basis Step: $3 \in S$
- Recursive Step: If $x \in S$ and $y \in S$, then $x + y \in S$.

Which of the following set-builder notations describes S?

◯ $\{x : x = 3k$ for some $k \in \mathbb{Z}, k \geq 0\}$
◯ $\{x : x = 3k$ for some $k \in \mathbb{Z}, k \geq 1\}$
◯ $\{x : x = 3 + k$ for some $k \in \mathbb{Z}, k \geq 0\}$
◯ $\{x : x = 3^k$ for some $k \in \mathbb{Z}, k \geq 1\}$

(f) Consider the following claim and rough proof and find the bug, ignoring any minor or stylistic issues.

Let Leaves(T) be the number of leaves in a binary tree T and Size(T) be the number of nodes a binary tree T.

Claim: For any non-empty binary tree T, Leaves$(T) \leq$ Size(T).

Proof:
Let $P(n)$ be "for all binary trees T of Size$(T) = n$, Leaves$(T) \leq n$". We prove $P(n)$ for $n \geq 1$ by induction.

Base Case ($n = 1$): A tree with 1 node is just a leaf. Leaves$(T) = 1$ and Size$(T) = 1$. $1 \leq 1$ is true. $P(1)$ holds.

Inductive Hypothesis: Assume $P(k)$ is true for some arbitrary $k \geq 1$.

Inductive Step: We want to show $P(k+1)$. Let T be a tree of size k. By the IH, Leaves$(T) \leq$ Size$(T) = k$. Now let T' be the tree T with one extra leaf node added. The size of this new tree T' is $k+1$.

When we added the leaf to get T', the number of leaves increases by at most 1. So, Leaves$(T') \leq$ Leaves$(T) + 1 \leq k+1$. Since Size$(T') = k+1$, we have Leaves$(T') \leq$ Size(T'). This proves $P(k+1)$.

Conclusion: Thus $P(n)$ is true for all $n \geq 1$.

What is the fundamental bug in this proof?

◯ The base case is wrong.
◯ The Inductive Hypothesis is stated incorrectly.
◯ You cannot use normal induction to prove a claim about trees.
◯ Showing Leaves$(T') \leq$ Size(T') does not show $P(k+1)$.
◯ The argument Leaves$(T) \leq$ Leaves$(T') + 1$ is incorrect.