CSE 311 : Practice Midterm 2 Exam Solutions

1. Proof by Contrapositive

(a) Prove the following claim using proof by contrapositive

For all primes p, if p is greater than 2, then $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$.

Hint 1: You may use without proof that all prime numbers greater than 2 are odd.
Hint 2: You may use the fact that every integer p must be congruent to 0, 1, 2, or 3 mod 4.
Hint 3: If your contrapositive assumption leads to $p = 4k$ or $p = 4k + 2$, remember that for p to be prime, its only *positive* divisors must be 1 and p. Showing p has another divisor (like 2 or 4) means it cannot be prime (unless p itself is 2).

Solution:

Let us prove this via contrapositive. The domain is all integers. The original claim is: $\forall p((\text{Prime}(p) \wedge p > 2) \rightarrow (p \equiv 1 \pmod 4) \vee p \equiv 3 \pmod 4)))$.

The contrapositive is: $\forall p(\neg(p \equiv 1 \pmod 4) \vee p \equiv 3 \pmod 4)) \rightarrow \neg(\text{Prime}(p) \wedge p > 2))$.

This simplifies to: $\forall p((p \not\equiv 1 \pmod 4) \wedge p \not\equiv 3 \pmod 4)) \rightarrow (\neg\text{Prime}(p) \vee p \leq 2))$.

Let p be an arbitrary integer and suppose $p \not\equiv 1 \pmod 4$ and $p \not\equiv 3 \pmod 4$. By the property of modular arithmetic, this means p must be either $p \equiv 0 \pmod 4$ or $p \equiv 2 \pmod 4$. We go by cases:

Case 1: $p \equiv 0 \pmod 4$.
By definition of congruence, $4 \mid p$. This means $p = 4k$ for some integer k. A divisor of p is 4. For p to be prime, its only divisors must be 1 and p (or -1 and $-p$). If $k = 0$, $p = 0$, which is not prime. If $k = 1$, $p = 4$, which is not prime (it has a divisor 2, which is not 1 or 4). If k is any other integer (e.g., 2, 3, -1, ...), p is a multiple of 4 and thus not prime (e.g., $p = 8$, $p = -4$). In all subcases, p is not prime, so the conclusion $(\neg\text{Prime}(p) \vee p \leq 2)$ is true.

Case 2: $p \equiv 2 \pmod 4$.
By definition of congruence, $4 \mid (p - 2)$. By definition of divides, $p - 2 = 4k$ for some integer k. So, $p = 4k + 2 = 2(2k + 1)$. Since k is an integer, $2k + 1$ is an integer. This shows that p has a divisor of 2. If $k = 0$, then $p = 2$. In this case, the conclusion $(\neg\text{Prime}(p) \vee p \leq 2)$ is true because $p \leq 2$. If $k \neq 0$, then p is an even number not equal to 2 (e.g., 6, -2, etc.). All such numbers are not prime (as they have a divisor of 2). Thus, $\neg\text{Prime}(p)$ is true, and the conclusion is true.

Since in all possible cases of the hypothesis ($p \equiv 0 \pmod 4$ or $p \equiv 2 \pmod 4$), the conclusion $(\neg\text{Prime}(p) \vee p \leq 2)$ holds, the contrapositive statement is true. Therefore, the original claim is true for all integers.

2. Set Theory

(a) Prove the following claim:

For all sets S and T, $\mathcal{P}(S) \cup \mathcal{P}(T) \subseteq \mathcal{P}(S \cup T)$.

Your proof must be in **English**. Do not write a logical equivalences proof. You can still use symbols within your **English proof** where appropriate.
Hint: Consider how the definition of union naturally creates cases (you can individually consider elements of the sets being unioned together). Then, see how the definition of powersets can be used to show elements of those sets are in $S \cup T$ and consider what this means for the powerset of $S \cup T$.

Solution:

Let S, T be arbitrary sets. Let X be an arbitrary element of $\mathcal{P}(S) \cup \mathcal{P}(T)$. By the definition of union, $X \in \mathcal{P}(S)$ or $X \in \mathcal{P}(T)$. Then there are 2 cases:

Case 1: $X \in \mathcal{P}(S)$. Then, by the definition of power set, $X \subseteq S$.
Let x be an arbitrary element of X. By definition of subset, $x \in S$.
Then certainly $x \in S$ or $x \in T$.
So, by definition of union, $x \in (S \cup T)$.
Since x was an arbitrary element of X, by the definition of subset, we have $X \subseteq (S \cup T)$.
Then, by definition of power set, $X \in \mathcal{P}(S \cup T)$.

Case 2: $X \in \mathcal{P}(T)$. Then, by the definition of power set, $X \subseteq T$.
Let x be an arbitrary element of X. By definition of subset, $x \in T$.
Then certainly $x \in S$ or $x \in T$.
By definition of union, $x \in (S \cup T)$.
Since x was an arbitrary element of X, by the definition of subset, we have $X \subseteq (S \cup T)$.
Then, by definition of power set, $X \in \mathcal{P}(S \cup T)$.

Since these cases are exhaustive, $X \in \mathcal{P}(S \cup T)$. Since X is arbitrary, by definition of subset, we've shown that $\mathcal{P}(S) \cup \mathcal{P}(T) \subseteq \mathcal{P}(S \cup T)$. Since S, T were arbitrary sets, the claim holds for all sets.

3. Induction

(a) Consider the following code snippet.

```
int Mystery(int n) {
    if n == 0:
        return 5
    if n == 1:
        return 16
    return 7 * Mystery(n - 1) - 10 * Mystery(n - 2)
}
```

In this problem, we will use `Mystery(n)` to refer to the value returned by the code snippet above when run on input `n`. For example, `Mystery(2)` $= (7 \cdot 16) - (10 \cdot 5) = 62$.

Use induction to show that `Mystery(n)` $= 3 \cdot 2^n + 2 \cdot 5^n$ for all integers $n \geq 0$. [20 points]

Solution:

Let $P(n)$ be "`Mystery(n)` $= 3 \cdot 2^n + 2 \cdot 5^n$." We will show $P(n)$ holds for all integers $n \geq 0$ by induction on n.

Base cases:
For $n = 0$, we have `Mystery(0)` $= 5 = 3 \cdot 1 + 2 \cdot 1 = 3 \cdot 2^0 + 2 \cdot 5^0$, so $P(0)$ holds.
For $n = 1$, we have `Mystery(1)` $= 16 = 6 + 10 = 3 \cdot 2 + 2 \cdot 5 = 3 \cdot 2^1 + 2 \cdot 5^1$, so $P(1)$ holds.

Inductive Hypothesis: Suppose $P(j)$ holds for all $0 \leq j \leq k$, for an arbitrary integer $k \geq 1$.

Inductive step: We will show that $P(k+1)$ holds, i.e. `Mystery(k + 1)` $= 3 \cdot 2^{k+1} + 2 \cdot 5^{k+1}$.

$$
\begin{aligned}
\texttt{Mystery(k + 1)} &= 7 \cdot \texttt{Mystery(k + 1 - 1)} - 10 \cdot \texttt{Mystery(k + 1 - 2)} && \text{Recursive case of Mystery; } k \geq 1 \text{ so } k+1 > 1 \\
&= 7(3 \cdot 2^k + 2 \cdot 5^k) - 10(3 \cdot 2^{k-1} + 2 \cdot 5^{k-1}) && \text{IH} \\
&= 21 \cdot 2^k + 14 \cdot 5^k - 30 \cdot 2^{k-1} - 20 \cdot 5^{k-1} && \text{Algebra} \\
&= 21 \cdot 2^k + 14 \cdot 5^k - 15 \cdot 2^k - 4 \cdot 5^k && \text{Algebra} \\
&= 6 \cdot 2^k + 10 \cdot 5^k && \text{Algebra} \\
&= 3 \cdot 2^{k+1} + 2 \cdot 5^{k+1} && \text{Algebra}
\end{aligned}
$$

Therefore, `Mystery(n)` $= 3 \cdot 2^n + 2 \cdot 5^n$ for all integers $n \geq 0$ by induction.

4. Proof by Contradiction

(a) Kuromi is buying boba for n friends. She notices that the total number of tapioca pearls, x, and the total price in cents, y, are related by the equation $x^2 - 2y^2 = 5$.

Prove by contradiction that at least one of the values, x or y, must be odd.

You may use the fact that an integer is even if and only if its square is even, and odd if and only if its square is odd.

Solution:

Let x and y be integers such that $x^2 - 2y^2 = 5$. We will prove the claim by contradiction.

The claim is Odd$(x) \vee$ Odd(y). Assume for the sake of contradiction that the claim is false. This means we assume $\neg$(Odd$(x) \vee$ Odd(y)). By De Morgan's Law, this is equivalent to $\neg$Odd$(x) \wedge \neg$Odd(y). This means we assume that x **is even** $\wedge$ y **is even**.

By definition of even: $x = 2k$ for some integer k. $y = 2j$ for some integer j.

Now, we substitute these into the given equation:

$$x^2 - 2y^2 = 5$$

$$(2k)^2 - 2(2j)^2 = 5$$
$$4k^2 - 2(4j^2) = 5$$
$$4k^2 - 8j^2 = 5$$

We can factor out a 2 from the left-hand side:

$$2(2k^2 - 4j^2) = 5$$

Let $z = 2k^2 - 4j^2$. Since k and j are integers, z must also be an integer. This gives us the equation $2z = 5$.

This implies that 5 is an even number (as it is 2 times some integer z). However, we know that 5 is an odd number.

This is a contradiction. Therefore, our original assumption (Even$(x) \wedge$ Even(y)) must be false.

It follows that the original claim (Odd$(x) \vee$ Odd(y)) must be true.

5. Multiple Choice Questions

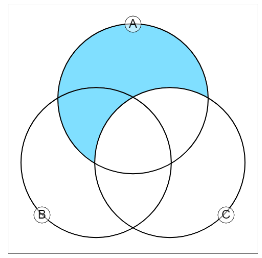
(a) Which of the following expressions represent the shaded area in the image? In the image, the top circle is A, the left circle is B, the right circle is C. **Select all that apply.** [3 points]

☐ $A \cap \overline{C}$

☐ $((A \cup B) \setminus C) \cap B$

☐ $\overline{A} \cap C$

☐ $((A \cup B) \setminus C) \cap A$



Solution:

Number 1, Number 4

(b) How many elements are in $(\{1, 2, 3\} \cup \{2, 4, 6\}) \setminus \{5, 6, 7\}$

◯ 0
◯ 3
◯ 4
◯ 9

Solution:

4, the set is $\{1, 2, 3, 4\}$

(c) Which of the following sets is the ordered pair $(1, 2)$ an element of? **Select all that apply**.

☐ $\mathbb{Z} \times \mathbb{R}$
☐ $\mathcal{P}(\{1, 2, 3\})$
☐ $\{(x, y) : x, y \in \mathbb{Z}, y > x\}$
☐ $\{1, 2, 3, 4, 5\}$

Solution:

First and third options

(d) Which of the following statements are true? **Select all that apply.**

Let $A = \{x \in \mathbb{Z} : x \equiv 1 \pmod 3\}$ Let $B = \{x \in \mathbb{Z} : x \equiv 1 \pmod 6\}$ Let $C = \{x \in \mathbb{Z} : x = 6k + 1$ for some $k \in \mathbb{Z}\}$ Let $D = \{1, 7, 13\}$

☐ $B \subseteq A$
☐ $A \subseteq C$
☐ $B \subseteq C$
☐ $D \subseteq B$

Solution:

$B \subseteq A$, $B \subseteq C$, and $D \subseteq B$ are all true.

(e) Consider the set S defined as:

- Basis Step: $3 \in S$
- Recursive Step: If $x \in S$ and $y \in S$, then $x + y \in S$.

Which of the following set-builder notations describes S?

◯ $\{x : x = 3k \text{ for some } k \in \mathbb{Z}, k \geq 0\}$
◯ $\{x : x = 3k \text{ for some } k \in \mathbb{Z}, k \geq 1\}$
◯ $\{x : x = 3 + k \text{ for some } k \in \mathbb{Z}, k \geq 0\}$
◯ $\{x : x = 3^k \text{ for some } k \in \mathbb{Z}, k \geq 1\}$

Solution:

The second option: $\{x : x = 3k \text{ for some } k \in \mathbb{Z}, k \geq 1\}$. (The set is $\{3, 6, 9, 12, \dots\}$, which are the positive multiples of 3.)

(f) Consider the following claim and rough proof and find the bug, ignoring any minor or stylistic issues.

Let Leaves(T) be the number of leaves in a binary tree T and Size(T) be the number of nodes a binary tree T.

Claim: For any non-empty binary tree T, Leaves(T) $\leq$ Size(T).

Proof:
Let $P(n)$ be "for all binary trees T of Size(T) $= n$, Leaves(T) $\leq n$". We prove $P(n)$ for $n \geq 1$ by induction.

Base Case ($n = 1$): A tree with 1 node is just a leaf. Leaves(T) $= 1$ and Size(T) $= 1$. $1 \leq 1$ is true. $P(1)$ holds.

Inductive Hypothesis: Assume $P(k)$ is true for some arbitrary $k \geq 1$.

Inductive Step: We want to show $P(k+1)$. Let T be a tree of size k. By the IH, Leaves(T) $\leq$ Size(T) $= k$. Now let T' be the tree T with one extra leaf node added. The size of this new tree T' is $k + 1$.

When we added the leaf to get T', the number of leaves increases by at most 1. So, Leaves(T') $\leq$ Leaves(T) $+ 1 \leq k + 1$. Since Size(T') $= k + 1$, we have Leaves(T') $\leq$ Size(T'). This proves $P(k + 1)$.

Conclusion: Thus $P(n)$ is true for all $n \geq 1$.

What is the fundamental bug in this proof?

◯ The base case is wrong.
◯ The Inductive Hypothesis is stated incorrectly.
◯ You cannot use normal induction to prove a claim about trees.
◯ Showing Leaves(T') $\leq$ Size(T') does not show $P(k + 1)$.
◯ The argument Leaves(T) $\leq$ Leaves(T') $+ 1$ is incorrect.

Solution:

The fourth option. The claim $P(k+1)$ says that "Leaves(T) $\leq$ Size(T) for all binary trees T of size $k + 1$". T' was of size $k+1$, but it was not arbitrary, so we can't make this claim about **all** binary trees of size $k+1$. A correct proof should have started with an **arbitrary** binary tree of size $k + 1$ and then reasoned about removing a node to use the inductive hypothesis on the resulting tree (or it could have used structural induction).