CSE 311 : Practice Midterm Exam Solutions

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Instructions

- You have eighty minutes to complete this exam.
- You are permitted one piece of 8.5x11 inch paper with handwritten notes (notes are allowed on both sides of the paper).
- You may not use a calculator or any other electronic devices during the exam.
- We will be scanning your exams before grading them. Please write legibly, and avoid writing up to the edge of the paper.
- **Problems are printed on both the front and back of each page!**
- You may also use the last page for extra space, but tell us where to find your answer if it's not right below the problem.
- If you want us to grade something you wrote on scratch paper, put your name and netid on the paper and tell us when you turn in your exam that you have an extra sheet.
- For multiple choice questions, options are shown in ◯ circles; completely fill in the circle for the (one) best answer. If options are shown in ☐ squares, completely fill in the squares for **ALL** correct answers.

Advice

- Remember to properly format English proofs (e.g. introduce all your variables).
- All proofs for this exam must be English proofs.
- If you don't initially know how to approach a proof, it's often helpful to write the start of the proof and put the "target" and conclusion at the bottom.
- Remember to take deep breaths.

1. Problem 1: Translation

For the rest of the page, let the domain of discourse be mammals. Interpret all sentences below as being in "mathematical English."

You may use the following predicates; the definition for the predicate is given after the colon in the list below.

- $\texttt{MountainGoat}(x)$: x is a mountain goat
- $\texttt{Sheep}(x)$: x is a sheep
- $\texttt{Climb}(x)$: x is climbing
- $\texttt{SummitsEverest}(x)$: x summits Mt. Everest
- $\texttt{ReachesTop}(x)$: x reaches the top
- $\texttt{Falls}(x)$: x falls
- $\texttt{Knows}(x, y)$: x knows y
- $\texttt{Strong}(x)$: x is strong

Translate the following English sentences into **PREDICATE LOGIC**.

(a) Every mountain goat that climbs and summits Mt. Everest is strong. **Solution:**

$\forall x((\text{MountainGoat}(x) \wedge \text{Climb}(x) \wedge \text{SummitsEverest}(x)) \rightarrow \text{Strong}(x))$

(b) For every mammal that summits Mt. Everest, there is a Mountain Goat that knows it. **Solution:**

$\forall x \exists y(\text{SummitsEverest}(x) \rightarrow (\text{MountainGoat}(y) \wedge \text{Knows}(y, x)))$

Write a **PREDICATE LOGIC** statement equivalent to the English statement below by taking the contrapositive of the implication inside it.

(c) For every mammal, if it falls, then it is a sheep.
Solution:

The original sentence in predicate logic is $\forall x(\text{Falls}(x) \rightarrow \text{Sheep}(x))$. Taking the contrapositive of the implication inside it yields $\forall x(\neg\text{Sheep}(x) \rightarrow \neg\text{Falls}(x))$.

Write the negation of the following sentences **IN ENGLISH**. Your English sentences must have negations applied only to individual predicates.

(d) There is a mountain goat that is climbing and reaches the top.
Solution:

The original sentence is $\exists x(\text{MountainGoat}(x) \wedge \text{Climb}(x) \wedge \text{ReachesTop}(x))$. The negation is $\forall x(\neg\text{MountainGoat}(x) \vee \neg\text{Climb}(x) \vee \neg\text{ReachesTop}(x))$. In English, this is "Every mammal is not a mountain goat or is not climbing or does not reach the top."

(e) There is a sheep that summits Mt. Everest that every mountain goat knows. **Solution:**

The original sentence is $\exists x \forall y(\text{Sheep}(x) \wedge \text{SummitsEverest}(x) \wedge (\text{MountainGoat}(y) \rightarrow \text{Knows}(y, x)))$. The negation is $\forall x \exists y(\neg\text{Sheep}(x) \vee \neg\text{SummitsEverest}(x) \vee (\text{MountainGoat}(y) \wedge \neg\text{Knows}(y, x)))$. In English: "For every mammal, either it is not a sheep or it does not summit Mt. Everest, or there exists a mountain goat that does not know it." A more natural phrasing is: "For every sheep that summits Mt. Everest, there is a mountain goat that does not know it."

2. Problem 2: Formal Inference Proof

Give a formal symbolic proof of the following statement.

$$\forall x \left[((\forall y\, P(x,y)) \leftrightarrow Q(x)) \rightarrow (Q(x) \rightarrow \exists y\, P(x,y)) \right]$$

Solution:

Let t be an arbitrary variable.

1.1. $(\forall y\, P(t,y)) \leftrightarrow Q(t)$	Assumption
1.2.1. $Q(t)$	Assumption
1.2.2. $[Q(t) \rightarrow (\forall y\, P(t,y))] \wedge [(\forall y\, P(t,y)) \rightarrow Q(t)]$	Definition of Biconditional: 1.1
1.2.3. $Q(t) \rightarrow (\forall y\, P(t,y))$	Elim $\wedge$: 1.2.2
1.2.4. $\forall y\, P(t,y)$	Modus Ponens: 1.2.1, 1.2.3
1.2.5. $P(t,c)$ for a fresh c	$\forall$ Elimination: 1.2.4
1.2.6. $\exists y\, P(t,y)$	$\exists$ Introduction: 1.2.5
1.2. $Q(t) \rightarrow (\exists y\, P(t,y))$	Direct Proof Rule
1. $((\forall y\, P(t,y)) \leftrightarrow Q(t)) \rightarrow (Q(t) \rightarrow \exists y\, P(t,y))$	Direct Proof Rule
2. $\forall x\, ((\forall y\, P(x,y)) \leftrightarrow Q(x)) \rightarrow (Q(x) \rightarrow \exists y\, P(x,y))$	$\forall$ Introduction: 1

3. Problem 3: Number Theory Proof

Show for all integers a, b, n where $n > 0$ that if $n|(b-a)$ then $n|[(b-2n)-(a+3n)]$.

In this problem, you **may** use the definitions of modular equivalence and divides, and algebra. You **may not** use other theorems about modular arithmetic unless you re-prove them.

Solution:

Let a, b, n be arbitrary integers with $n > 0$. Suppose $n|(b-a)$.

By the definition of divides, there exists some integer k such that $nk = b - a$.

We want to show that $n|[(b-2n)-(a+3n)]$. By definition, this means we need to show that $nk = (b-2n) - (a+3n)$.

Let's rearrange the expression $(b-2n)-(a+3n)$:

$$\begin{aligned}(b-2n)-(a+3n) &= b - 2n - a - 3n \\ &= (b-a) - 5n \\ &= nk - 5n \quad \text{(by substituting our earlier result)} \\ &= n(k-5)\end{aligned}$$

Since k is an integer, $k-5$ is also an integer. Let $j = k-5$. Then we have shown that $(b-2n)-(a+3n) = nj$ for some integer j.

Therefore, by the definition of divides, this means $n|((b-2n)-(a+3n))$.

4. Problem 4: Short Answer

(a) Which of the following describes what it means when we write $\exists x \forall y P(x, y)$?

◯ All the values (x) in our domain have the same value y that makes $P(x, y)$ true. (y cannot depend on x)
◯ All the values (y) in our domain have a value x that makes $P(x, y)$ true. (x can depend on y).
◯ There is a value (x) in our domain so that for all values y, $P(x, y)$ is true. **Solution:**

> The third option is correct.

(b) Which of the following expressions are equivalent to $(p \vee q) \wedge v$? Mark ALL that apply.

☐ $(\neg p \rightarrow q) \wedge v$
☐ $(q \rightarrow \neg p) \wedge v$
☐ $(p + q) \cdot v$
☐ $(p \rightarrow q) \wedge v$
☐ $p \vee (q \wedge v)$
Solution:

> The first and third options are correct.

(c) Which of the following expressions are **NOT** equivalent to $(p \vee q) \vee r$? Mark ALL that apply.

☐ $r \rightarrow \neg(p \wedge q)$
☐ $p \vee (q \vee r)$
☐ $(\neg p \rightarrow q) \vee r$
☐ $p \vee (\neg q \rightarrow r)$
☐ $\neg(p \vee q) \rightarrow r$
Solution:

> The first option is the only one that is not equivalent. $r \rightarrow \neg(p \wedge q) \equiv \neg r \vee \neg p \vee \neg q$.

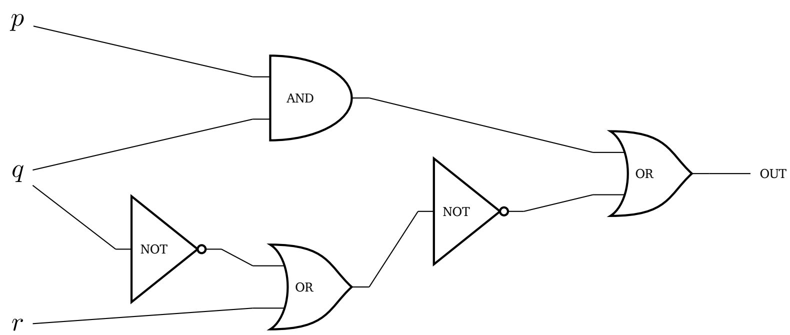
(d) Which of the following is the **DNF** of the given truth table?

p	q	r	Output
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

◯ $(p \vee q \vee r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee r)$
◯ $(p \wedge q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$
◯ $(\neg p \wedge \neg q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r)$
◯ $(\neg p \vee \neg q \vee \neg r) \wedge (p \vee \neg q \vee \neg r) \wedge (p \vee q \vee \neg r)$
Solution:

> The second option is correct. The DNF is constructed from the rows where the output is T.

(e) Which of the following is the circuit below equivalent to?



◯ $(p \wedge \neg q) \vee \neg(\neg q \vee r)$
◯ $(p \wedge q) \vee (\neg q \vee r)$
◯ $(p \wedge q) \vee \neg(\neg q \vee r)$
◯ $(p \wedge q) \wedge \neg(\neg q \vee r)$

Solution:

The third option is correct.

(f) You wish to show "For every integer, if it is divisible by 10 then it is even" with a proof by contrapositive. State the contrapositive of the claim in English. **Solution:**

For every integer, if it is odd, then it is not divisible by 10.

(g) A reliable source tells you the following statement is true: "If a student is taking 311, then they know DeMorgan's Law." What can you conclude about the statement "If a student knows DeMorgan's Law, then they are taking 311." ?

◯ The second statement must be true.
◯ The second statement cannot be true.
◯ The second statement might or might not be true.

Solution:

The third option is correct. The second statement is the converse, which is not logically equivalent to the original statement.