CSE 311 : Practice Final Solutions

This exam is a (slight) modification of a real final given in a prior quarter of CSE311.

The original exam was given in a 110 minute slot.

We strongly recommend you take this exam as though it were closed book – even though your exam will be open book.

Instructions

- Students had 110 minutes to complete the exam.
- The exam was closed resource (except for the logical equivalences, boolean algebra, and inference rules reference sheets). Your exam will allow a notes sheet.
- The problems are of varying difficulty.
- If you get stuck on a problem, move on and come back to it later.

1. Aquatic Translation [10 points]

Translate the following sentences into logical notation if the English statement is given or to an English statement if the logical statement is given, taking into account the domain restriction. Let the domain of discourse be sea creatures and seas. Use predicates Fish and Sea to do the domain restriction. You can use $\texttt{SwimIn}(x, y)$ which is true if and only if x can swim in y. The predicate $\texttt{NarwhalsLive}(x)$ is true if and only if narwhals live in x, and $\texttt{ContainsIcebergs}(x)$ is true if and only if x contains icebergs.

(a) Every fish can swim in some sea.

Solution:

$\forall x \exists y (\texttt{Fish}(x) \rightarrow [\texttt{Sea}(y) \land \texttt{SwimIn}(x, y)])$

(b) There is a fish that cannot swim in every sea.

Solution:

$\exists x \forall y (\texttt{Fish}(x) \land [\texttt{Sea}(y) \rightarrow \neg \texttt{SwimIn}(x, y)])$

(c) Some fish can only swim in one sea.

Solution:

$\exists x \exists y (\texttt{Fish}(x) \land \texttt{Sea}(y) \land \texttt{SwimIn}(x, y) \land \forall z [(\texttt{Sea}(z) \land \texttt{SwimIn}(x, z) \rightarrow (y = z)])$

(d) $\forall x [(\texttt{Sea}(x) \land \texttt{NarwhalsLive}(x)) \rightarrow \texttt{ContainsIcebergs}(x)]$

Solution:

Every sea where narwhals live contains icebergs.

(e) $\exists x (\texttt{Sea}(x) \land \texttt{NarwhalsLive}(x) \land \texttt{ContainsIcebergs}(x) \land \forall y [(\texttt{Sea}(y) \land \texttt{NarwhalsLive}(y)) \rightarrow (x = y)])$

Solution:

There is only one sea where narwhals live and that sea contains icebergs. OR, The only sea where narwhals live contains icebergs.

2. Regularly Irregular [15 points]

Let $\Sigma = \{0, 1\}$. Prove that the language $L = \{x \in \Sigma^* : \#_0(x) < \#_1(x)\}$ is irregular.

Solution:

Let D be an arbitrary DFA. Consider $S = \{0^n : n \geq 0\}$. Since S is infinite and D has finitely many states, we know $0^i \in S$ and $0^j \in S$ both end in the same state for some $i < j$. Append 1^j to both strings to get:

$a = 0^i 1^j$ Note that $a \in L$, because $i < j$ and $0^i 1^j \in \Sigma^*$.

$b = 0^j 1^j$ Note that $b \notin L$, because $j \not< j$.

Since a and b both end in the same state, and that state cannot both be an accept and reject state, D cannot recognize L. Since D was arbitrary, no DFA recognizes L; so, L is irregular.

3. Recurrences, Recurrences [15 points]

Define

$$T(n) = \begin{cases} n & \text{if } n = 0, 1 \\ 4T\left(\lfloor \frac{n}{2} \rfloor\right) + n & \text{otherwise} \end{cases}$$

Prove that $T(n) \leq n^3$ for $n \geq 3$.

Solution:

We go by strong induction on n. Let $P(n)$ be "$T(n) \leq n^3$" for $n \in \mathbb{N}$.

Base Cases. When $n = 3$, $T(3) = 4T\left(\lfloor \frac{3}{2} \rfloor\right) + 3 = 4T(1) + 3 = 7 \leq 27 = 3^3$.
When $n = 4$, $T(4) = 4T\left(\lfloor \frac{4}{2} \rfloor\right) + 4 = 4T(2) + 4 = 28 \leq 64 = 4^3$.
When $n = 5$, $T(5) = 4T\left(\lfloor \frac{5}{2} \rfloor\right) + 5 = 4T(2) + 5 = 29 \leq 4^4$.

Induction Hypothesis. Suppose $P(3) \wedge P(4) \wedge \cdots \wedge P(k)$ for some $k \geq 5$.

Induction Step. We want to prove $P(k+1)$: Note that

$$\begin{aligned}
T(k+1) &= 4T\left(\left\lfloor \frac{k+1}{2} \right\rfloor\right) + k + 1, && \text{because } k+1 \geq 2. \\
&\leq 4\left(\left\lfloor \frac{k+1}{2} \right\rfloor\right)^3 + k + 1, && \text{by IH.} \\
&\leq 4\left(\frac{k+1}{2}\right)^3 + k + 1, && \text{by def of floor.} \\
&= 4\left(\frac{(k+1)^3}{2^3}\right) + k + 1, && \text{by algebra.} \\
&= \frac{(k+1)^3}{2} + k + 1, && \text{by algebra.} \\
&= \frac{(k+1)((k+1)^2 + 2)}{2}, && \text{by algebra.} \\
&\leq \frac{(k+1)((k+1)^2 + (k+1)^2)}{2}, && \text{because } (k+1)^2 \geq 2. \\
&= (k+1)^3, && \text{by algebra}
\end{aligned}$$

Thus, since the base case and induction step hold, the $P(n)$ is true for $n \geq 3$.

4. Structural CFGs [15 points]

Consider the following CFG: **S** $\to \varepsilon$ | **SS** | **S**1 | **S**01. Another way of writing the recursive definition of this set, Q, is as follows:

- $\varepsilon \in Q$
- If $S \in Q$, then $S1 \in Q$ and $S01 \in Q$
- If $S, T \in Q$, then $ST \in Q$.

Prove, by structural induction that if $w \in Q$, then w has at least as many 1's as 0's.

Solution:

We go by structal induction on w. Let $P(w)$ be "$\#_0(w) \leq \#_1(w)$" for $w \in \Sigma^*$.

Base Case. When $w = \varepsilon$, note that $\#_0(w) = 0 = \#_1(w)$. So, the claim is true.

Induction Hypothesis. Suppose $P(w), P(v)$ are true for some w, v generated by the grammar.

Induction Step 1. Note that $\#_0(w1) = \#_0(w) \leq \#_1(w) + 1 = \#_1(w1)$ by IH, and $\#_0(w01) = \#_0(w) + 1 \leq \#_1(w) + 1 = \#_1(w01)$ by IH.

Induction Step 2. Note that $\#_0(wv) = \#_0(w) + \#_0(v) \leq \#_1(w) + \#_1(v)$ by IH.

Since the claim is true for all recursive rules, the claim is true for all strings generated by the grammar.

5. Tralse! [15 points]

For each of the following answer True or False and give a short explanation of your answer.

(a) Any subset of a regular language is also regular. **Solution:**

False. Consider $\{0,1\}^*$ and $\{0^n 1^n : n \geq 0\}$. Note that the first is regular and the second is irregular, but the second is a subset of the first.

(b) The set of programs that loop forever on at least one input is decidable. **Solution:**

False. If we could solve this problem, we could solve HaltNoInput. Intuitively, a program that solves this problem would have to try all inputs, but, since the program might infinte loop on some of them, it won't be able to.

(c) If $\mathbb{R} \subseteq A$ for some set A, then A is uncountable. **Solution:**

True. Diagonalization would still work; alternatively, if A were countable, then we could find an surjective function between $\mathbb{N}$ and $\mathbb{R}$ by skipping all the elements in A that aren't in $\mathbb{R}$.

(d) If the domain of discourse is people, the logical statement

$$\exists x\ (P(x) \rightarrow \forall y\ (x \neq y \rightarrow \neg P(y))$$

can be correctly translated as "There exists a unique person who has property P". **Solution:**

False. Any x for which $P(x)$ is false makes the entire statement true. This is not the same as there existing a unique person with property P.

(e) $\exists x\ (\forall y\ P(x,y)) \rightarrow \forall y\ (\exists x\ P(x,y))$ is true regardless of what predicate P is. **Solution:**

True. The left part of the implication is saying that there is a single x that works for all y; the right one is saying that for every y, we can find an x that depends on it, but the single x that works for everything will still work.

6. Relationships and Sets! [15 points]

(a) Let $S = \{(X, Y) : X, Y \in \mathcal{P}(\mathbb{N}) \wedge X \subseteq Y\}$.

Recall that R is antisymmetric iff $((a, b) \in R \wedge a \neq b) \rightarrow (b, a) \notin R$.

Prove that S is antisymmetric.

Solution:

Suppose $(a, b) \in S$ and $a \neq b$. Then, by definition of S, $a \subset b$ and there is some $x \in b$ where $x \notin a$ (since they aren't equal). Then, $(b, a) \notin S$, because $b \not\subseteq a$, because $x \in b$ and $x \notin a$. So, S is antisymmetric.

(b) Let $A = \{x : x \equiv k(\mod 4)\}$, $B = \{x : x = 4r + k \text{ for some integer } r\}$. Prove that $A = B$ for all integers k.

Solution:

Let k be an arbitrary integer.

Let x be an arbitrary element of A, so $x \in A$. By definition of A, $x \equiv k(\mod 4)$. By definition of mod, $4|k - x$. By definition of divides, there exists an integer r such that $4r = k - x$. Rearranging, we see $x = k - 4r = k + 4(-r)$. Since x was arbitrary, $A \subseteq B$.

Let y be an arbitrary element of B, so $y \in B$. Then $y = 4r + k$ for some integer r. Rearranging, we get $4r = y - k$. By definition of divides, $4|y - k$. By definition of mod, $y \equiv k(\mod 4)$. Thus by definition of A, $y \in A$. Since y was arbitrary, $B \subseteq A$.

Because $A \subseteq B$ and $B \subseteq A$, then $A = B$.

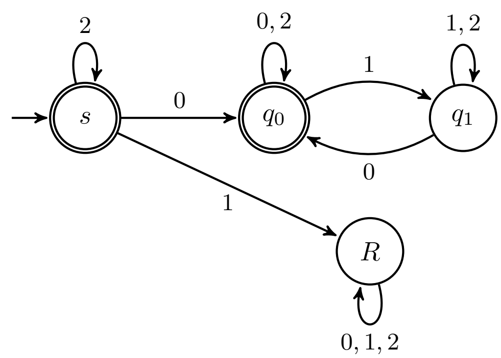
7. All The Machines! [15 points]

Let $\Sigma = \{0, 1, 2\}$.
Consider $L = \{w \in \Sigma^* :$ Every 1 in the string has at least one 0 before and after it$\}$.
Note that the 0s need not be directly adjacent to the 1s.

(a) Give a regular expression that represents A. **Solution:**

$(0 \cup 2)^*(0(0 \cup 1 \cup 2)^*0)^*(0 \cup 2)^*$

(b) Give a DFA that recognizes A. **Solution:**



(c) Give a CFG that generates A.

Solution:

$$S \to 0S \mid 2S \mid S2 \mid 0T0 \mid \varepsilon$$
$$T \to 0T \mid 1T \mid 2T \mid \varepsilon$$

(d) Let $\Sigma = \{0, 1, 2\}$. Construct a DFA that recognizes exactly strings with a 0 in all positions i where $i\%3 = 0$.

Solution: