Homework 2: Circuits, Boolean Algebra, and More Logic

Due date: Wednesday October 8th at 11:59 PM
If you work with others (and you should!), remember to follow the collaboration policy outlined in the syllabus.
In general, you are graded on your work's clarity and accuracy. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.
We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we expect. You can have longer explanations, but explanations significantly longer than necessary may receive deductions.

Be sure to read the grading guidelines on the assignments page.

0. Math Background [7 points]

In a few weeks, we'll start doing algebraic manipulations on numbers; we'll need rules for manipulating exponents and summations. Generally, these are things you learned at some point, but you might be a little rusty on. We have a few example multiple choice questions on this gradescope assignment. The problem is set up like a concept check (explanations will appear when you're right, and you can resubmit as many times as you'd like).

1. Proof [23 points]

In Lecture 3 we gave a symbolic proof that $(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (\neg a \vee b)$. In this problem, we will give another proof.

(a) Our intuition for the proof in class was "the last two pieces of the formula correspond to vacuous truth (when a is false)." Identify a commonality in the first two pieces of the formula and describe it.
(Your description should be similar in spirit to the one from class, but it's ok if you don't have a technical term like 'vacuous truth') [4 points]

(b) Give another proof of the formula that matches the intuition from part a instead of the intuition from class. [16 points]
Read the symbolic proof guidelines on the assignments page before you start.
Hint: your proof, if it matches your intuition from (a) will be different from the one from class – at least some of the intermediate expressions will be different.

(c) In class we labeled portions of the proof in purple with high-level descriptions of what they are doing (lecture 3 slide 44, left side). Produce similar labels for your proof. Submit your answer in the form "Steps [X] to [Y]: [label]." [3 points]
Note: The goal here is to give intuition for what is happening at a higher level than individual steps.

2. Circuit du Soleil [10 points]

In this problem, we'll construct two propositions in terms of the variables p, q, r and then use these propositions to build a circuit that computes a binary function $M(p, q, r)$.

(a) Give a propositional logic formula containing only the variables q and r which evaluates to false when q is true and evaluates to $\neg r$ when q is false. [2 points]

(b) Give a propositional logic formula containing only the variables p and q which evaluates to p when q is true and evaluates to true when q is false. [2 points]

(c) Now consider the binary function $M(p, q, r)$ which is defined as:

$$M(p, 0, r) := \neg r$$

$$M(p, 1, r) := \neg p$$

Draw a circuit that takes p, q, r as input, uses only AND, OR, and NOT gates, and outputs $M(p, q, r)$. Your gates should not take more than two inputs.
Your answer for this part **must** combine your answers from (a) and (b)! [6 points]

3. 'Cause baby, now we got Contrapositives! [15 points]

(a) The song sounds good if it has piano.

  (i) Convert this sentence to propositional logic (as on homework 1, ensure you're assigning variables to **atomic propositions**, not compound ones). [2 points]

  (ii) Take the contrapositive symbolically, and simplify so that any $\neg$ signs are next to atomic propositions (i.e. only single variables). You are not required to show work for this part. [2 points]

  (iii) Translate the contrapositive back to English. [3 points]

  (iv) Compare your English sentence from (iii) to the original implication. Do they mean the same thing? (Just say "yes" or "no" here) [0.5 points]

(b) Isaiah will play the game Silksong, only if he has time and it's not windy outside.

Repeat steps (i)-(iv) from (a) for this sentence.

4. Two of a Kind [20 points]

(a) Translate the Boolean Algebra expression $[(X + Y) \cdot (X + Y)']' \cdot [X' + ((X + Y) \cdot (X + Y'))]$ to Propositional Logic. Use the variables x and y to represent the propositions $X = 1$ and $Y = 1$, respectively. [2 points]

(b) Prove that your solution to (a) is a tautology using a chain of equivalences. [16 points]

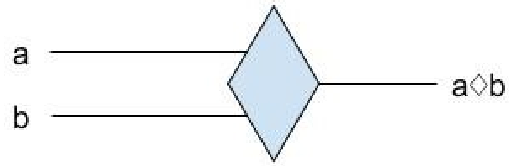
(c) Why do we know that the Boolean Algebra expression from part (a) will always evaluate to 1? Explain (1-2 sentences). [2 points]

5. Shine Bright Like a Diamond [12 points]

Consider a new logical operator $\diamond$ (*diamond* in latex). For this problem, we define $a \diamond b$ by the following truth table:

A	B	$A \diamond B$
1	1	0
1	0	1
0	1	1
0	0	1

Even though this is a new operator, we can still use it to create circuits representing other operators we have seen before! Here is an example drawing of the diamond gate:



(a) Using only $\diamond$ gates and the input a, create a circuit whose output represents $\neg a$. You may use multiple copies of a as inputs if required. [4 points]

(b) Using only $\diamond$ gates and the inputs a and b, create a circuit that's output represents $a \wedge b$. You may use multiple copies of a, b as inputs if required. [4 points]
Hint You might find it helpful for this problem to start by drawing a truth table with both $a \diamond b$ and $a \wedge b$.

(c) Using only $\diamond$ gates and the inputs a and b, create a circuit whose output represents $a \vee b$. You may use multiple copies of a, b as inputs if required. [4 points]

6. A Tale of Two For-Alls [12 points]

Consider the following two expressions:

$$\exists x(\mathsf{P}(x) \wedge \mathsf{Q}(x)) \qquad \exists x(\mathsf{P}(x)) \wedge \exists x(\mathsf{Q}(x))$$

(a) Give one domain of discourse and definitions of P and Q such that these expressions **are not** equivalent. Explain why your examples work (1-2 sentences). [6 points]

(b) Give one domain of discourse and definitions of P and Q such that these expressions **are** equivalent. Explain why your examples work (1-2 sentences). [6 points]

7. In the Real World: Legal-ese [12 points]

Computer scientists aren't the only people who need to precisely interpret long English sentences. Lawyers and judges need to interpret long (but, hopefully, precise) sentences in law codes. These groups sometimes run into the same types of problems we have experienced in interpreting these sentences. In fact, a U.S. Supreme Court case in 2024 involved a disagreement on how to interpret a statement that could have been (maybe *should* have been) put in propositional logic.

The case was about a provision in a particular law, giving lighter sentences to defendants with less criminal history. It said (in part)

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(*) A defendant is eligible for relief [a lighter sentence], if the defendant does not have—
 (a) Condition A;
 (b) Condition B; and
 (c) Condition C.
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Where conditions A, B, and C were specific measures of prior criminal convictions. A specific defendant (Pulsifer) did not have Condition C, but did have Conditions A and B. So, could he get a lighter sentence?

Pulsifer said he could! If we let e be the proposition "the defendant is eligible for relief", and a, b, c be propositions corresponding to the conditions in the law, then the law said $[\neg(a \wedge b \wedge c)] \rightarrow e$. Since c is false for him, the hypothesis is true, meaning that he must be eligible for relief.

But prosecutors said he was not necessarily eligible. They thought the law said $[\neg a \wedge \neg b \wedge \neg c] \rightarrow e$.

Take a moment to look at the English sentence at issue (the sentence in `monospace` after the (*) above) and look the two interpretations in propositional logic. Notice what the interpretations have in common: both think we have an implication, both combine a, b, c with ANDs, but they disagree on how "not" relates to the conditions: does it apply to the combination of all three, or does it relate individually to each one? Try to put yourself in each party's mindset and see how the English sentence could plausibly mean what they claim.

If you don't feel like you understand the two interpretations, you might want to read this case summary or Section II of the Supreme Court's opinion, which have more details on the law.

(a) At the Supreme Court argument, Pulsifer's lawyer said "The government [prosecutors] needs 'AND' to mean 'OR'", but in propositional logic, the prosecutors still have "AND"s.
Convert the prosecutor's propositional logic to replace the $\wedge$s with $\vee$s and show your work with a chain of equivalencies (citing the appropriate rules). For this part, you are allowed to apply associativity, commutativity, and double negation without explicitly writing the step. [4 points]
Hint: If you've done it right, the structure should look a lot like Pulsifer's interpretation of the law, just with OR's instead of AND's: $\neg(a \vee b \vee c) \rightarrow e$. That is to say, if you agree with where Pulsifer's lawyer put the parentheses, the alternate interpretation does want "AND" to mean "OR."

(b) Try to write a better English sentence than Congress did—write one that can correspond to Pulsifer's interpretation **only**, not the prosecutors' interpretation. You probably will want to use additional words. Phrases like "simultaneously" or "everything in this list" or "at least one thing in this list" can help make a sentence like this more clear; you might also try moving the "not" to be closer to each of the conditions. Alternatively, taking a contrapositive could help! [3 points]

(c) Do the same as the previous part but for the prosecutors' interpretation. Again, adding words or taking the contrapositive may be helpful to express the logic more clearly in English. [3 points]

(d) Take a moment to appreciate having propositional logic! It's hard to write an English sentence conveying either interpretation that can't be confused with the other one. But notation lets us differentiate between the two (comparatively) easily. You do not have to write anything for this part [0 points]

(e) What do you think is the "right" interpretation of the law? Pulsifer's? The prosecutors'? Is the law ambiguous with no clear interpretation? What should a court do in a situation like this? We won't grade your answer on whether we agree with you, just whether you've addressed the questions (3-4 sentences, but feel free to write more).[1] [2 points]

8. Feedback [1 point]

Answer these questions on the separate gradescope box for this question.

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment (excluding any extra credit questions, if applicable)? Report your estimate to the nearest hour.
- Which problem did you spend the most time on?
- Any other feedback for us?

[1] Many expert lawyers and judges have come down on both sides of this argument, there's plenty of room for disagreement!