Binary Trees

Basis: A single node is a rooted binary tree.

Recursive Step: If $T_1$ and $T_2$ are rooted binary trees with roots $r_1$ and $r_2$, then a tree rooted at a new node, with children $r_1, r_2$ is a binary tree.

size($\bullet$) = 1
size($T_1$) + size($T_2$) + 1

height($\bullet$) = 0
height($T_1$) + height($T_2$) = 1 + max(height($T_1$), height($T_2$))

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Structural Induction Template

1. Define $P()$ State that you will show $P(x)$ holds for all $x \in S$ and that your proof is by structural induction.

2. Base Case: Show $P(b)$
   [Do that for every $b$ in the basis step of defining $S$]

3. Inductive Hypothesis: Suppose $P(x)$
   [Do that for every $x$ listed as already in $S$ in the recursive rules].

4. Inductive Step: Show $P()$ holds for the “new elements.”
   [You will need a separate step for every element created by the recursive rules].

5. Therefore $P(x)$ holds for all $x \in S$ by the principle of induction.
Functions on Strings
Since strings are defined recursively, most functions on strings are as well.

Length:
\[ \text{len}(\varepsilon) = 0; \]
\[ \text{len}(wa) = \text{len}(w) + 1 \text{ for } w \in \Sigma^*, a \in \Sigma \]

Reversal:
\[ \varepsilon^R = \varepsilon; \]
\[ (wa)^R = aw^R \text{ for } w \in \Sigma^*, a \in \Sigma \]

Concatenation
\[ x \cdot \varepsilon = x \text{ for all } x \in \Sigma^*; \]
\[ x \cdot (wa) = (x \cdot w)a \text{ for } w \in \Sigma^*, a \in \Sigma \]

Number of c's in a string
\[ \#_c(\varepsilon) = 0 \]
\[ \#_c(wa) = \#_c(w) + 1 \text{ for } w \in \Sigma^*; \]
\[ \#_c(wa) = \#_c(w) \text{ for } w \in \Sigma^*, a \in \Sigma \setminus \{c\}. \]

Claim for all \( x, y \in \Sigma^* \) \( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \). 

Let \( P(y) \) be “\( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x \in \Sigma^* \).”

We prove \( P(y) \) for all \( x \in \Sigma^* \) by structural induction.

Base Case:
Inductive Hypothesis
Inductive Step:

We conclude that \( P(y) \) holds for all string \( y \) by the principle of induction. Unwrapping the definition of \( P \), we get \( \forall x \forall y \in \Sigma^* \, \text{len}(xy) = \text{len}(x) + \text{len}(y) \), as required.