Proof By Contradiction

In real life!

- Claim: My Tire is Leaking
- Suppose that this tire was not leaking
- This means the tire pressure should be constant
- I observe the pressure is dropping at a moderate rate
- But there should be constant pressure if it was not leaking
- Therefore, it must be leaking

Proof by Contradiction Skeleton

Claim: p is true.

- Suppose for the sake of contradiction $\neg p$.
- …

- Then some statement s must hold.
- …

- And some statement $\neg s$ must hold.
- But s and $\neg s$ is a contradiction. So p must be true.

My tire is leaking

Suppose my tire is not leaking

The tire pressure must be constant

The tire pressure is decreasing

My Tire is leaking

Why does this work?

Let's say the claim you are trying to prove is p.

A proof by contradiction shows the following implication:

$$\neg p \to False$$

Why does this implication show p?

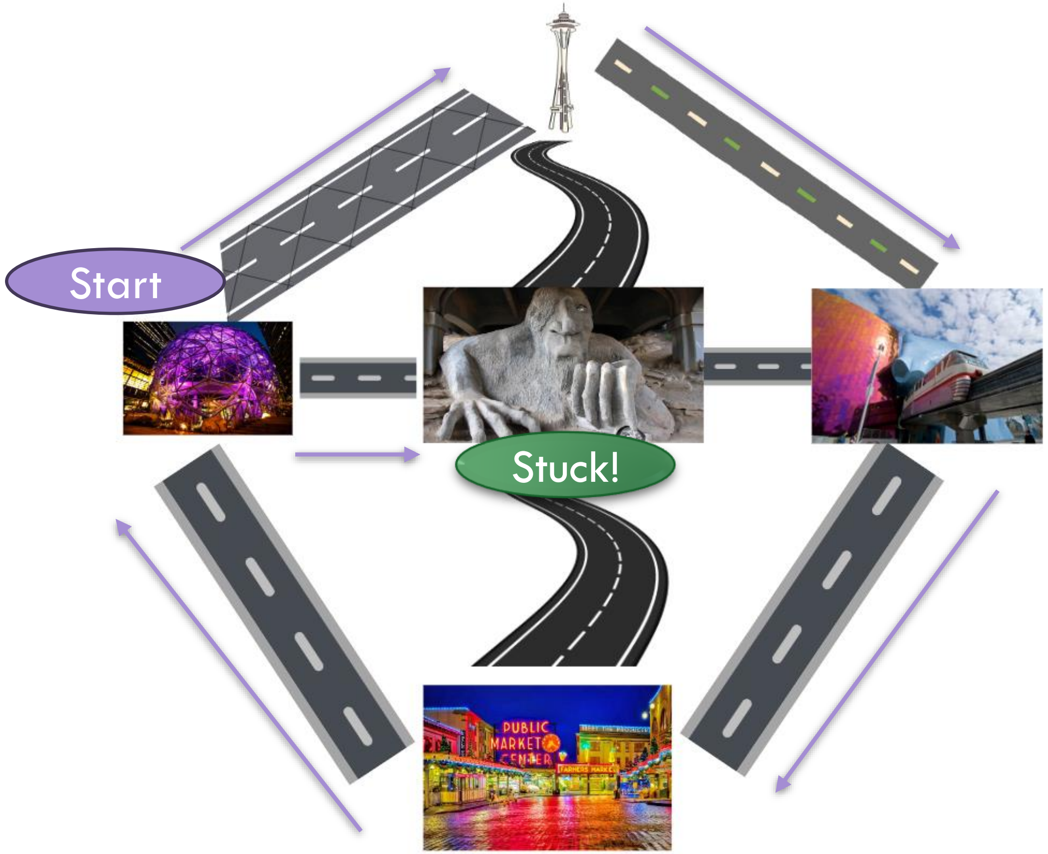
Hint think contrapositive

The contrapositive is $True \to p$ which simplifies to just p.

This means that by proving $\neg p \to False$, you have proved p is True!

Graph Example

Can we travel on every road, without going on a road twice?



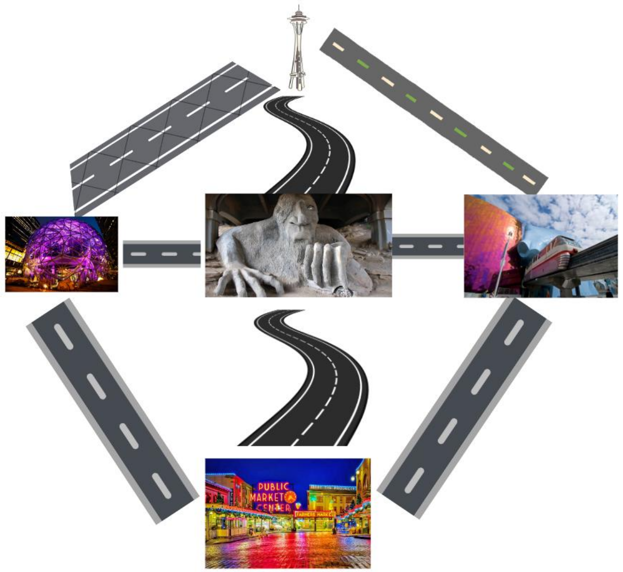
There is no path, let's prove it!

Graph Example

Claim: it is impossible to travel on every road visiting each road exactly once

Proof: Suppose that it is *possible* to travel on every road visiting each road exactly once.

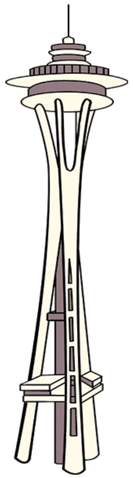
Consider how many times each vertex would be passed through on this path.



However [] is a contradiction!

Therefore, it must be impossible to visit every road exactly once

We enter and exit a landmark



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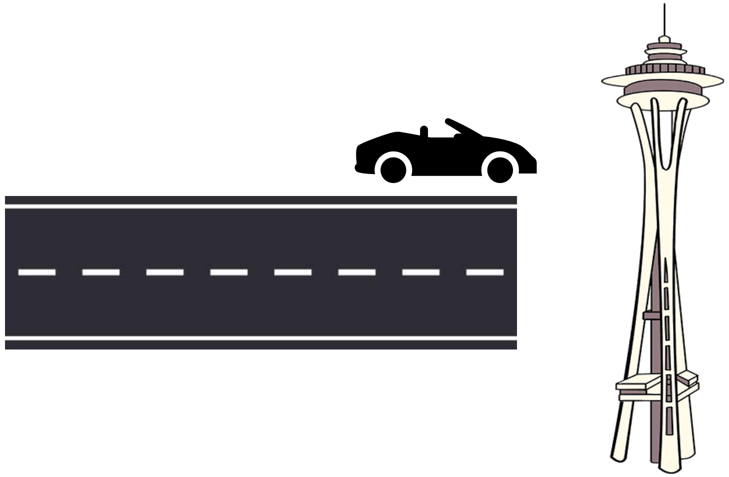

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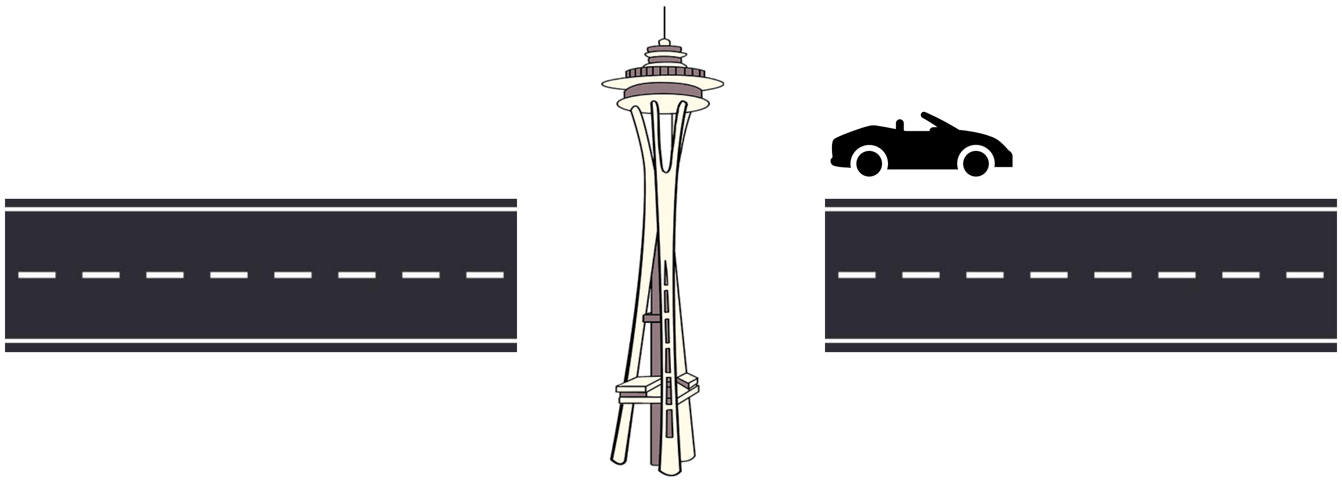
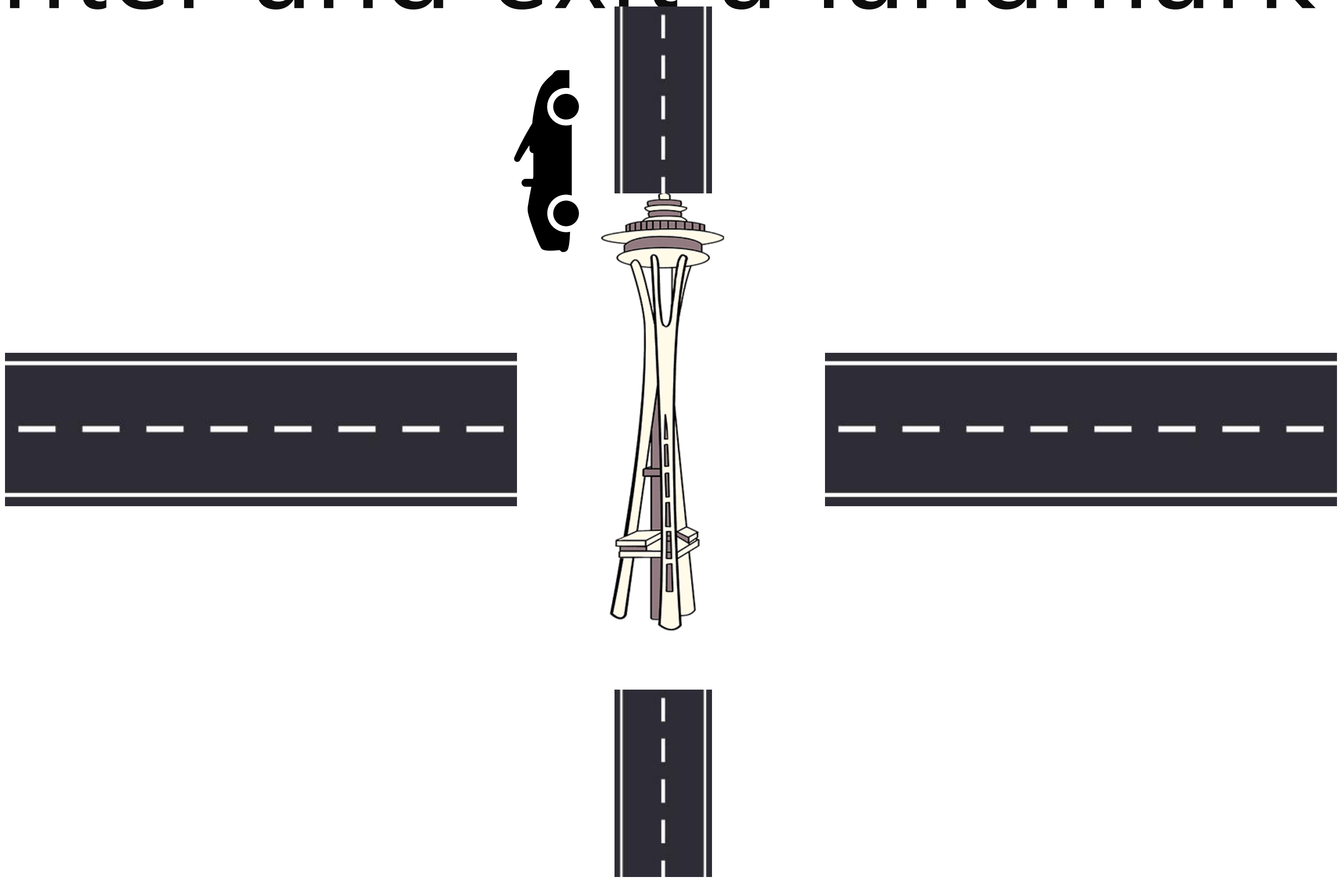


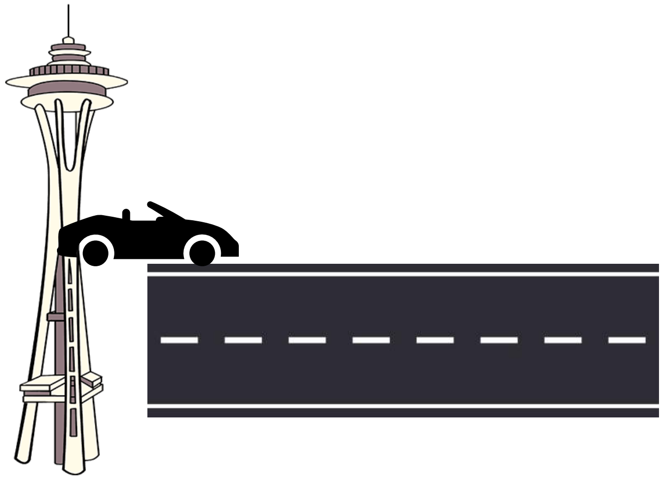
Notice that this means there are an even number of roads that we drove on connected to this landmark

We enter and exit a landmark



Even if we go through it again on new roads, this holds

We Start at the Landmark



Notice we drove on only one road, (as we *started* in the landmark) making it have an odd number of roads that connect to it

We End at the Landmark

Notice we drove on only one road, (as we *ended* in the landmark) making it have an odd number of roads that connect to it

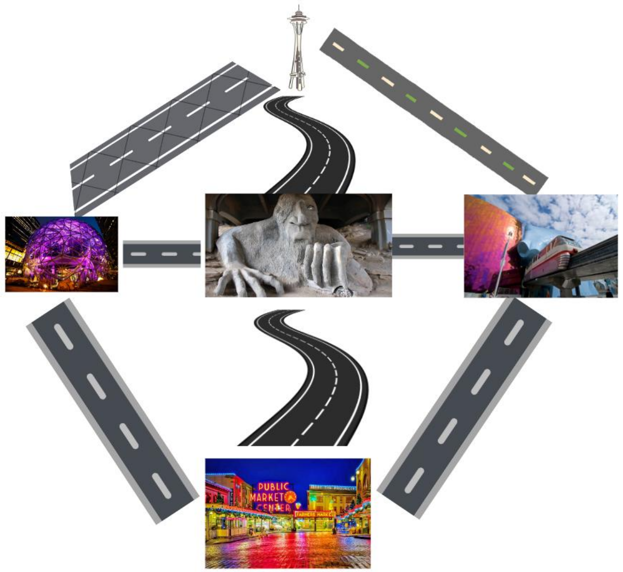
Graph Example

Claim: it is impossible to travel on every road visiting each road exactly once

Proof: Suppose that it is *possible* to travel on every road visiting each road exactly once.

Consider how many times each landmark would be passed through on this path.

As we observed, all of the landmarks on our path must have an even number of roads, except for the starting and ending one, making us have exactly 2 landmarks with an odd number of connecting roads.



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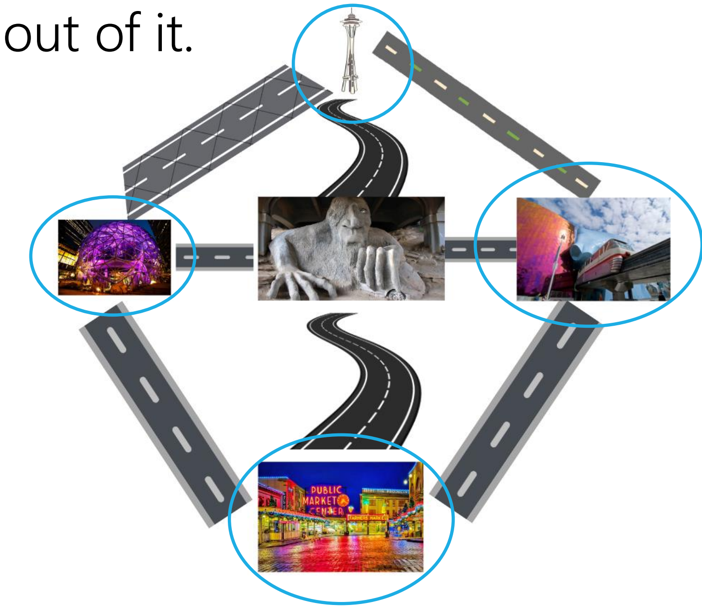
However, our graph has 4 landmarks with an odd number of roads coming out of it.



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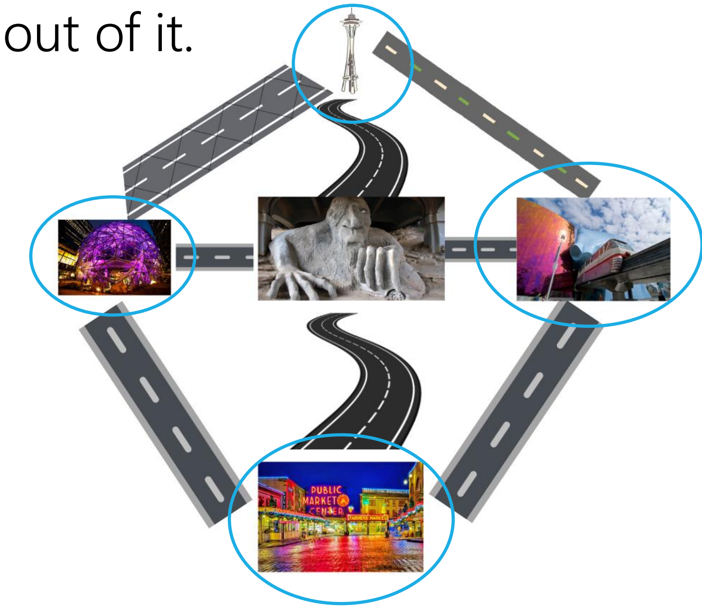

But since 2 is not 4, this is a contradiction!

Therefore, it must be impossible to visit every road exactly once

Proof by Contradiction Examples

Proof By Contradiction

If a^2 is even, then a is even

Claim: $\sqrt{2}$ is irrational (i.e not rational)

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Suppose for the sake of contradiction that $\sqrt{2}$ is rational

Notice target is unknown

But [] is a contradiction! Thus, we can conclude that $\sqrt{2}$ is irrational.

Proof By Contradiction

If a^2 is even, then a is even

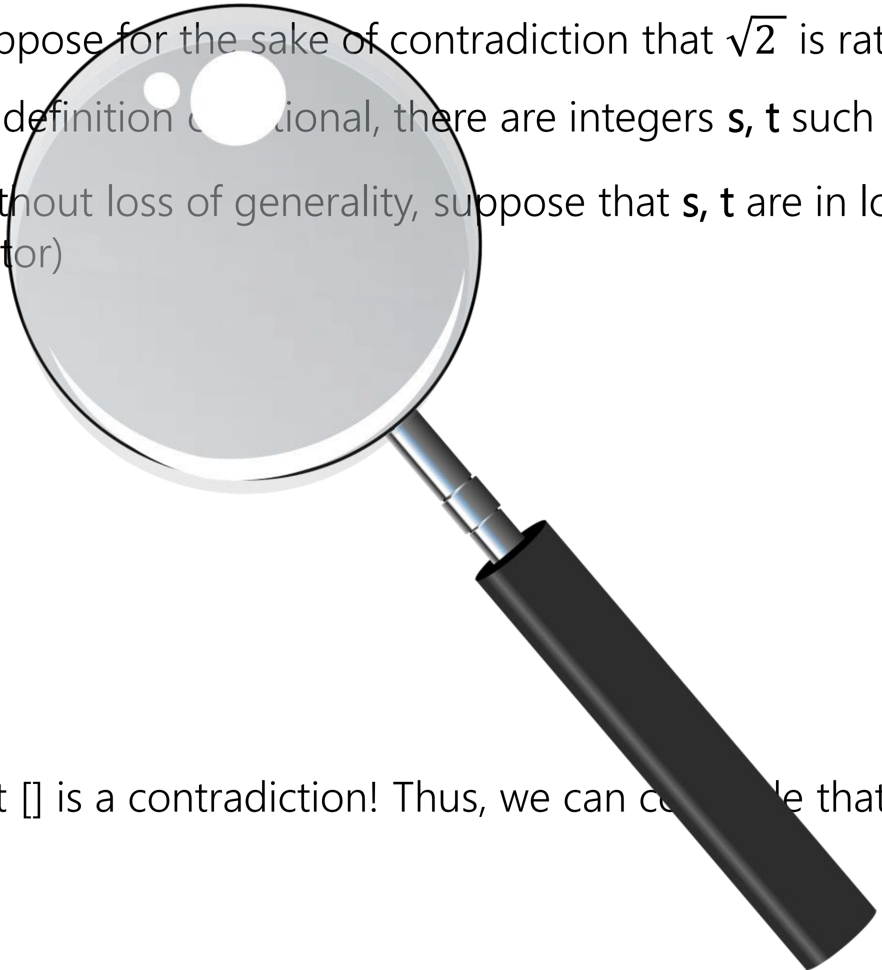
Claim: $\sqrt{2}$ is irrational (i.e not rational)

Proof:

Suppose for the sake of contradiction that $\sqrt{2}$ is rational.

By definition c tional, there are integers **s, t** such that $t \neq 0$ and $\sqrt{2} = \frac{s}{t}$

Without loss of generality, suppose that **s, t** are in lowest terms (i.e it is the reduced fraction and 1 is **s, t** greatest common factor)



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What is "Without Loss of Generality"?

You can use this when it looks like you are introducing a new assumption, but you are not, and the claim is still general. Only use if it would be immediately obvious to the reader why it is the case

In this case: if **s** and **t** share a factor other than 1, i.e k, we can just cancel out their common factor and continue the proof. (i.e $\frac{s\prime k}{t\prime k} = \frac{s}{t}$)

Another example:

Let x,y be integers; without loss of generality, assume $x \geq y$.

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Suppose for the sake of contradiction that $\sqrt{2}$ is rational.

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Thus: $2t^2 = s^2$ So s^2 is even, making s even by our lemma. This means that $s = 2k$ for some integer k

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Dividing both sides by two, we get $\mathrm{t}^2 = 2k^2$, making t^2 is even, making t even by our lemma.

But if both s and t are even, they must have a common factor of 2. But we said that the fraction $\frac{s}{t}$ was irreducible.

This is a contradiction! Thus, we can conclude that $\sqrt{2}$ is irrational.

Proof by Contradiction

Proof by contradiction is a strategy for proving statements of any form.

- The general strategy to prove p is to assume $\neg p$ and derive False.

Examples:

- The strategy to prove $p \to q$ is to assume $p \land \neg q$ and derive False.
- The strategy to prove $p \lor q$ is to assume $\neg p \land \neg q$ and derive False.
- The strategy to prove $\forall x(P(x))$ is to assume $\exists x\big(\neg P(x)\big)$ and derive False.
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Another Proof by Contradiction

Claim: There are infinitely many primes

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Suppose for the sake of contradiction, there are only finitely many primes. Call them $p_1, p_2, \dots, p_k$.

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Suppose for the sake of contradiction, there are only finitely many primes. Call them $p_1, p_2, \dots, p_k$.

Where can we find a contradiction?

- Show our list is non inclusive (i.e create a different prime number)
- Show one of the numbers in our list is not prime
- Create a contradiction with facts about prime factorization
- Show 1 = 2
- Show p is odd and even at the same time
- Proof by cases with a mix of the above

But [] is a contradiction! So, there must be infinitely many primes.

Proof by Contradiction: Remarks

- Unlike other proof techniques, we don’t know *where* we’re going.
  We’re trying to find **any** contradiction. That can make it harder.
- Contradiction is a sledge-hammer.
  It can be used to prove many things. But it makes a mess.
- You can find a contradiction directly with your assumption

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Suppose for the sake of contradiction, there are only finitely many primes. Call them $p_1, p_2, \ldots, p_k$.

Consider the number $\mathrm{q} = p_1 \cdot p_2 \cdot \ldots \cdot p_k + 1$

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Case 2: q is not prime (i.e composite):

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Notice that q is prime and must be larger that every prime in $p_1, p_2, \ldots, p_k$. But every prime was in the list, therefore this is a contradiction!

Case 2: q is not prime (i.e composite):

Since q is composite, we know that some prime p_i must divide q. This means that $q \;\%\; p_i = 0$.

Also, notice that $\mathrm{q} \;\%\; p_i = (p_1 \cdot p_2 \cdot \ldots \cdot p_k) + 1 \;\%\; p_i$ using the definition of q, which gives us:

$$\mathrm{q} \;\%\; p_i = (p_1 \cdot p_2 \cdot \ldots \cdot p_k) + 1 \;\%\; p_i = (p_1 \cdot \ldots p_i \cdot \ldots \cdot p_k) + 1 \;\%\; p_i = 1$$

This means that $q \;\%\; p_i$ equals both 1 and 0, which is impossible!

In both cases, this is a contradiction! So, there must be infinitely many primes.

Bonus Proof!

Claim: if a^2 is even, than a is even.

Proof:

Suppose for the sake of contradiction that a^2 is even *and* a is odd for some integer a.

This means that $\mathrm{a} = 2k + 1$ for some k.

Substituting this in, we have $a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$

Since $2k^2 + 2k$ is an integer, we have that a^2 is odd!

This is a contradiction however as a^2 cannot be both even and odd. Therefore through proof by contradiction, if a^2 is even, than a is even.