# Homework 7: Structural Induction, Regexes, CFGs

**Version 2**: Updated 2/23 1:30 AM. Fixed small typo in questions 1 and 3
Due date: Wednesday February 28th at 11:59 PM
If you work with others (and you should!), remember to follow the collaboration policy outlined in the syllabus.

In general, you are graded on both the clarity and accuracy of your work. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.

We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we are expecting. You are allowed to have longer explanations, but explanations significantly longer than necessary may receive deductions.

To help with formatting of English proofs, we've published a style guide on the website containing some tips. **Unless otherwise noted in a problem, all proofs must be English proofs.** You should not have numbered steps (e.g., you should not be doing an inference proof.)

Finally, be sure to read the grading guidelines for more information on what we're looking for.

## 1. Walk the walk, talk the talk [20 points]

Let $S$ be a subset of $\mathbb{Z} \times \mathbb{Z}$ defined recursively as:

**Basis Step:** $(1, 0) \in S$ and $(0, 1) \in S$
**Recursive Step:** if $(a, b) \in S$ and $(c, d) \in S$, then $(a, b) + (c, d) = (a + c, b + d) \in S$.

We claim that $S = (\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}) \setminus \{(0, 0)\}$; that is, $S$ is the set of integer coordinates in the upper right quadrant (excluding the origin). It is easy to show that any such point is in $S$: if $(x, y)$ is in this quadrant, then $x, y \geq 0$, meaning that:

$$(x, y) = x(1, 0) + y(0, 1) = (1, 0) + \cdots + (1, 0) + (0, 1) + \cdots + (0, 1)$$

where we add $x$ copies of $(1, 0)$ and $y$ copies of $(0, 1)$. This point is in $S$ by the additive recursive step. To show that points not in the upper right quadrant are not in the set $S$, we will prove the following claim by structural induction.

**Prove:** For every $(a, b) \in S$, we have $a \geq 0$ and $b \geq 0$, but also $a > 0$ or $b > 0$.

## 2. From Exploding Snap to Chinese Fireball [23 points]

You and Ron Weasley are sitting in the Gryffindor common room and are bored of playing Exploding Snap. You decide to open a new game that you bought at Weasleys' Wizard Wheezes called Chinese Fireball. In this game there is a miniature Chinese Fireball dragon who's health is represented by an ordered pair of two integer values $(w, n)$ where the first number represents how much additional fire the dragon requires, and the second represents how many additional meals it needs. The ideal state for a dragon is at $(0, 0)$. When it reaches this point, the dragon matures and begins to fly.

The dragon starts off at some state $(k, k)$ and you and Ron take turns playing the game. In their turn, each player can either cast incendio to add fire to the dragon (decreasing the amount of fire it needs to hatch) or conjure up some bacon to feed the dragon (decreasing the amount of meals it needs to hatch), but a player can only do one of those two options (not both). The player gets to pick how much they affect the dragon (but its state must change by at least 1)

More formally, assuming you are at state $(a, b)$ you can move to any of the states: $(a, b - i)$ where $1 \leq i \leq b$ or $(a - j, b)$ where $1 \leq j \leq a$.

A player wins the game by being the one who causes the dragon to hit $(0, 0)$, as that player gets to keep this dragon as a pet!

You generously allow Ron to go first.

(a) Using induction, prove that in any game of Chinese Fireball where the dragon starts in state $(a, a)$ you (the player that goes second) can always win the game.

Be sure to explicitly and clearly define a predicate $P()$! We **strongly** recommend your predicate includes the phrase "It is not my turn" or "the second player can" or something similar. The predicate you define should only take in one input. [20 points]

(b) Describe your winning strategy (i.e. describe how you should play the game in order to win, assuming that you go **SECOND**). A strategy would be something like "If my friend casts incendio on the dragon for $i$ units then I will..." [3 points]

## 3. Prove or Disprove [20 Points]

For each of the following problems, either prove the above statement or disprove the above statement with a counter-example.

(a) Prove or Disprove that for arbitrary sets $A$, $B$, that $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$

(b) Prove or Disprove that for arbitrary sets $A$, $B$, that $\mathcal{P}(A \setminus (A \cap B)) = \mathcal{P}(A \cap \overline{B})$

## 4. Constructing Regular Expressions (Online) [15 points]

For each of the following languages, construct a regular expression that matches exactly the given set of strings.

You will submit (and check!) your answers online at https://grin.cs.washington.edu/. Think carefully before entering a submission; you only have 10 guesses. Because these are auto-graded, we will not award partial credit.

(a) Strings over $\{0, 1, 2\}$ where every $2$ comes (directly) before a $0$ or $1$.

(b) Binary strings with at least two 1's or at most two 0's.

(c) Binary strings that start with 0 and have length congruent to 2 (mod 3).

(d) **Extra Credit:** Strings over $\{0, 1, 2\}$ that start with $0$ and have odd length, where ever occurrence of a $2$ is followed (directly) after by a $0$ or $1$ [0 Points].

## 5. Context Is Everything. Except for Context-Free Grammars (Online) [10 points]

For each of the following languages, construct a context-free grammar that generates exactly the given language.

You will submit (and check!) your answers online at https://grin.cs.washington.edu/. Think carefully before entering a submission; you only have 10 guesses. Because these are auto-graded, we will not award partial credit.

(a) All strings over $\{0, 1, 2\}$ of the form $x2y$, with $x, y \in \{0, 1\}^*$ and $y$ a subsequence of $x^R$ (i.e., it is $x^R$ with some characters possibly removed), where $x^R$ is defined as the reverse of $x$.

Note: "Subsequence" is not the same thing as "substring".

(b) All strings in the form of $\{a^x b^y a^{2x+y} : x, y \geq 0\}$

(c) **Extra Credit:** The set of all binary strings with an odd number of 0's and an even number of 1's [0 Points].

## 6. Feedback [1 point]

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment?
- Which problem did you spend the most time on?
- Any other feedback for us?