Homework 6: More Induction

Version 2: Updated 2/14 6 PM. Question 5 has major changes to definitions of tree, leaves, and height.

Due date: Wednesday February 21st at 11:59 PM
If you work with others (and you should!), remember to follow the collaboration policy outlined in the syllabus.

In general, you are graded on both the clarity and accuracy of your work. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.

We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we are expecting. You are allowed to have longer explanations, but explanations significantly longer than necessary may receive deductions.

In order to assist with writing English proofs, we've published a style guide on the website containing some tips. This guide contains references to proof materials that we haven't taught yet, so don't worry if some of these terms are unfamiliar.

Finally, be sure to read the grading guidelines for more information on what we're looking for.

1. Contra-⟨?⟩ [10 points]

Write a proof by contradiction for the statement "For all integers a and b: if $a \nmid b$ or $b \nmid a$, then $a \neq b$".

Note that $x \nmid y$ means that "x does not divide y", which is the negation of $x \mid y$.

2. Set Proof [10 points]

Let A, B, and C be arbitrary sets. Prove that $(B \cap C) \setminus A \subseteq (B \setminus A) \cap (C \setminus A)$.

You **MUST** use induction for problems 3-5 unless the directions for a part say otherwise.
You may use any appropriate version of induction (e.g. weak or strong or structural).
Remember to define a predicate $P()$ as part of your proof.

3. The Proof is in the Law of Implication [23 points]

(a) Show that given three logical propositions a, b and c that

$$a \to (b \to c) \equiv (a \wedge b) \to c.$$

This should be a simplification style equivalence proof where each line cites an equivalence. [3 points]

(b) Show using induction for an integer $n \geq 3$, given n logical propositions $a_1, a_2, \cdots a_{n-1}, a_n$, that

$$a_n \to (a_{n-1} \to (\cdots \to (a_2 \to a_1) \cdots)) \equiv (a_n \wedge a_{n-1} \wedge \cdots \wedge a_2) \to a_1.$$

Hint: For this problem, you will want to use regular induction with a simplification style equivalence proof where each line cites an equivalence in the IS. You might also want to reference part (a) in your proof. [20 points]

4. Alligator Eats The Bigger One [20 points]

Prove that for all integers n with $n \geq 1$ we have $3^n + 1 \leq 3^{n+1} - 3^{n-1}$.

5. The Leaves Don't Fall Far From The... `Tree` [20 points]

In CSE123, you saw a recursive definition of trees. That definition looks a little different from what we saw in class.

The following definition is analogous to what you saw in 123. For this problem, we will consider binary trees defined as follows:

Basis Step: $(\texttt{null}, \bullet, \texttt{null})$ is a `tree`.
Recursive Step: If L, R are `trees` then $(L, \bullet, R)$ is also a `tree`

We will also use the following recursively defined functions for leaves:

$$\texttt{leaves}((\texttt{null}, \bullet, \texttt{null})) = 1$$
$$\texttt{leaves}((L, \bullet, R)) = \texttt{leaves}(L) + \texttt{leaves}(R) \qquad \text{for two arbitrary } \texttt{trees } L, R$$

And height:

$$\texttt{height}((\texttt{null}, \bullet, \texttt{null})) = 0$$
$$\texttt{height}((L, \bullet, R)) = 1 + \max(\texttt{height}(L), \texttt{height}(R)) \qquad \text{for two arbitrary } \texttt{trees } L, R$$

Show that for all `trees` t: $\texttt{leaves(t)} \leq 2^{\texttt{height}(t)}$