Homework 4: English Proofs

Version 2: Updated 1/25 8 PM. Fixed small typo in question 1
Due date: Wednesday, January 31th at 11:59 PM
If you work with others (and you should!), remember to follow the collaboration policy outlined in the syllabus.
In general, you are graded on both the clarity and accuracy of your work. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.
We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we are expecting. You are allowed to have longer explanations, but explanations significantly longer than necessary may receive deductions.

Be sure to read the grading guidelines on the assignments page for more information on what we're looking for.

1. Why spiders? Why couldn't it have been follow the butterflies? [16 points]

Translate each of the following sentences into predicate logic notation. These translations require some of our quantifier tricks.
Note: For each part, your answer should include each predicate (for that part) at least once.

(a) **Domain**: Hogwarts's students
Predicates: $\texttt{FirstYear}(x), \texttt{SecondYear}(x), \texttt{EnterForbiddenForest}(x), \texttt{InGraveDanger}(x)$

All first and second year students who enter the Forbidden Forest are in grave danger. [4 points]

(b) **Domain**: Creatures in the forbidden forest
Predicates: $\texttt{Spider}(x), \texttt{ChasingHarryAndRon}(x), \texttt{Equals}(x, y)$

There are at least two spiders chasing Harry and Ron. [4 points]

(c) **Domain**: Creatures in the Forbidden Forest
Predicates: $\texttt{Centaur}(x), \texttt{LookingOutForHarryAndRo}(x), \texttt{Equals}(x, y)$

There are only two centaurs looking out for Harry and Ron. [4 points]

(d) **Domain**: Potion ingredients in the Forbidden Forest and potions in the Wizarding World
Predicates: $\texttt{Potion}(x), \texttt{InAdvancedPotionMakingBook}(x), \texttt{PotionIngredient}(x), \texttt{IngredientFor}(x, y)$
(Note that $\texttt{IngredientFor}(x, y)$ returns true when a potion y requires an ingredient x.)

Every potion from the Advanced Potion-Making book has at least one ingredient that can be found in the Forbidden Forest. [4 points]

2. Being Direct [12 points]

Let the domain of discourse for this problem be integers. Define the predicates $\text{Odd}(x) := \exists k(x = 2k + 1)$, and $\text{Even}(x) := \exists k(x = 2k)$.

(a) Translate the following claim to predicate logic: [4 points]

For all odd integers n and m, $3n - m$ is even.

(b) Prove that the claim is true. [8 points] For this problem, write an English proof (like we did in Lecture 8, for example)

3. Oddly Even [12 points]

In this problem we will analyze the statement:

For all integers n, if $n - 3$ is even, then n^2 must be odd.

(a) Translate the claim into predicate logic. Let your domain of discourse be integers, and use the definitions in the previous problem. [4 points]

(b) Prove the claim is true. [8 points] Again, do an English proof.

4. Mary, Mary, Quite Contrary [10 points]

When direct proofs fail, our logical equivalences can come to the rescue. Consider the statement

For every integer k, if $k^3 + 5$ is odd then k is even

Proving this directly is not easy (try it for yourself to see!). Instead, we will prove the contrapositive of this statement.

(a) Write the contrapositive of the given statement (in English). [2 points]

(b) Write a proof by contrapositive (do an English proof) of the given statement. [8 points]

5. Something is wrong here... [12 points]

5.1. Ice scream, you scream, we all scream for ICE CREAM!! [6 points]

It's 9 PM on Wednesday, your 311 homework is submitted, and it's time to scour the Ave for a sweet treat. The cold temperatures aren't enough to dissuade you from getting ice cream, so you stop at Sweet Alchemy.

Assume the following things to be true.

- Sweet Alchemy has a sale going on! You notice that the ube ice cream is discounted.
- Your dessert tonight is going to be on a budget (since we're college students and food these days is too expensive)
- The flavors available tonight at Sweet Alchemy are ube, taro, durian, and banana (quite the interesting selection!)
- Your dessert tonight will be some ice cream that you purchase during your stop at Sweet Alchemy.

Consider the following (incorrect) proof of the proposition "Tonight's dessert must be ube ice cream!".

1. Your dessert was purchased at Sweet Alchemy tonight.
2. Ube, taro, durian, and banana were the only four flavors available at the store the entire day.
3. We know that tonight's dessert must be purchased on a budget.
4. Ube ice cream is on sale.
5. So we conclude that tonight's flavor is ube.

(a) Identify the most significant error in the proof and discuss why this step is incorrect. Sentences have been labeled to easily refer back to specific portions of the proof.

Note: This homework is **NOT** sponsored by Sweet Alchemy.

5.2. What's wrong with this proof? [6 points]

Consider the following statement:

For all real numbers a, b, and c, if $ac = bc$, then $a = b$.

And the following spoof (incorrect proof) of the statement:

> Let a, b, c be arbitrary real numbers and suppose that $ac = bc$. Dividing both sides of the equation by c, we obtain $\frac{ac}{c} = \frac{bc}{c}$ which simplifies to $a = b$. Thus, our claim holds.

(a) Why is the above proof incorrect?

Here, let's try again. This must be correct this time, right?

> Let a and b be arbitrary real numbers and suppose that $a = b$. Let c be an arbitrary real number. Multiplying both sides of the equation by c, we obtain $ac = bc$.

(b) Again, why is the above proof incorrect?

(c) Is the original statement true or false? If the statement is true, write a correct proof. If it is false, provide a counterexample.

6. Divides [8 points]

Write an English proof to show that if 8 divides $(x + 5)$ (i.e. $8|(x + 5)$) for an integer x, then x is odd. Recall that English proofs don't have domains of discourse, so you need to state the types for your variables when you introduce them.

7. Galleons! [Extra Credit]

You will submit this question to the separate gradescope box for "homework 4 extra credit."

Figure 1: A niffler sitting on gold coins.

Six nifflers found a stash of 100 galleons. The nifflers are coincidentally named Anna, Jacob, Owen, Robbie, Sam and Timothy. They agreed to split the galleons using the following rules:

- The first niffler in alphabetical order becomes the leader of the nifflers.
- The leader of the nifflers proposes how to split the galleons. For example, they might say "Anna gets 100 galleons, all other nifflers get 0." or "Anna, Jacob, Owen, Robbie, and Sam get 20 galleons, Timothy gets none."

- All nifflers (including the leader) vote for or against the proposal.
  - If 2 or more nifflers disagree to the proposal, Tina Goldstein will cast stupefy on the leader of the nifflers so that the leader will be stunned. Once stunned, the leader no longer participates in the splitting process (they do not vote, and they cannot receive galleons in the split).
  - Otherwise, the galleons will be split as proposed.

Thus, the first round Anna is the leader: if her proposal has been rejected by at least 2 nifflers, she be stunned and Jacob becomes the leader, etc; If Anna, Jacob, Owen, Robbie, and Sam get stunned, then Timothy will become the leader and keep all the galleons.

The nifflers' first priority is not to be stunned, since being stunned makes them sad. If they don't get stunned, they will try to get as many galleons as possible for themselves, as they are greedy and attracted to shiny things. Finally, they don't like each other, so if they can get the same amount of galleons for agreeing and disagreeing with the proposal, they will disagree with the proposal and cause the leader to be stunned.

Assuming that all 6 nifflers are smart (and greedy and all aware of the others' intelligence and greediness), what will happen?

Your solution should indicate which nifflers will be stunned, and how many galleons each of the remaining nifflers will receive.

8. Feedback [1 point]

Answer these questions on the separate gradescope box for this question.

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment (excluding any extra credit questions, if applicable)? Report your estimate to the nearest hour.
- Which problem did you spend the most time on?
- Any other feedback for us?