Homework 3: Predicate Logic

Version 2: Updated 1/18 3 PM. Added clarification for question 2 (guidelines for domain of discourse)

Due date: Wednesday January 24th at 11:59 PM
If you work with others (and you should!), remember to follow the collaboration policy outlined in the syllabus.
In general, you are graded on both the clarity and accuracy of your work. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.
We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we are expecting. You are allowed to have longer explanations, but explanations significantly longer than necessary may receive deductions.

Be sure to read the grading guidelines on the assignments page for more information on what we're looking for.

1. Animals and Rock Climbing

We've written some sentences below about some of the staff's favorite things: animals and rock-climbing.

Let the domain of discourse be all mammals. Use the following definition of these predicates for this problem:

- $\texttt{MountainGoat}(x)$: x is a mountain goat
- $\texttt{Sheep}(x)$: x is a sheep
- $\texttt{Climb}(x)$: x is climbing (Treat "climbs" and "is climbing" as equivalent ideas in this problem)
- $\texttt{SummitsEverest}(x)$: x summits Mt. Everest
- $\texttt{Strong}(x)$: x is strong
- $\texttt{ReachesTop}(x)$: x reaches the top

1.1. Round One [12 points]

Translate the following observations into English. Your translations should take advantage of "restricting the domain" to make more natural translations when possible, but you should not otherwise simplify the expression before translating. Specifically, we have these requirements for translations in this problem

- You must not use variable names in your English translation (e.g., don't say "for every x...")
- For every quantified variable where one or more of the predicates can be interpreted as a domain restriction, you must use at least one of them to make your translation more natural. So with a domain of discourse of all integers, $\forall x([\texttt{Even}(x) \land \texttt{Prime}(x)] \rightarrow \texttt{IsEqual}(x, 2))$ could be translated as "For every even integer, if it is prime it is equal to 2" or "Every prime and even integer is equal to 2" but could not be translated as "For every integer, if the integer is prime and even then it is equal to 2."
- If the sentence does not have domain restriction, you may use "every mammal" or "some mammal" to refer to an arbitrary element of the domain.

(a) $\exists x\,(\texttt{Sheep}(x) \land \neg\,\texttt{Climb}(x) \land \texttt{Strong}(x))$

(b) $\forall x(\texttt{MountainGoat}(x) \rightarrow \neg\,\texttt{Climbing}(x)) \land \forall x(\texttt{Sheep}(x) \rightarrow (\texttt{Strong}(x) \land \texttt{ReachesTop}(x)))$

(c) $\neg\forall x((\texttt{MountainGoat}(x) \land \texttt{Climb}(x)) \rightarrow \texttt{SummitsEverest}(x))$

1.2. Round Two [4 points]

You realize that the first sentence (i.e., part a) is false. State the negation of (a) in English. You should simplify the negation so that the English sentence is natural. Simplify here mainly means that negations should be applied only to individual predicates. For example you must say "x is not prime and it is not even" rather than "it is not the case that x is prime or x is even."

2. Become a Domain Expert [10 points]

For the following statements, translate them into predicate logic (specifying and defining any predicates you use). Then provide a domain of discourse where the statement is true and another domain of discourse where the statement is false.

Also include 1-2 sentences for each domain for why the statement has the truth value it does.

If you wish to make extra assumptions about the world you may do so.

(a) For every x and y, there is a z such that z is the sum of x and y. Broadly, both domains of discourse you give should be some group of numbers.

(b) There is a x who will do y's laundry for every y who has dirty laundry. Broadly, both domains of discourse you give should be some group of people.

3. Nested Quantifiers [15 points]

Fix your domain of discourse to be all real numbers.

Use the predicates $\texttt{Natural}(x)$, $\texttt{Integer}(x)$, and $\texttt{Prime}(x)$ to say x is a natural number, integer, or prime, respectively. Similarly, use the predicates $\texttt{Positive}(x)$ and $\texttt{Non-Negative}(x)$ to say x is positive or non-negative. Finally, the predicate $\texttt{LessThan}(x, y)$ means $x < y$ (note the order).

In this problem, an example of something you might give for a "scenario" might be "There is at least one number in the domain of discourse that is not prime". You can assume that the given statements are true facts about numbers.

(a) Your friend tried to translate "Every natural number is non-negative and an integer" and got

$$\forall x(\texttt{Natural}(x) \wedge [\texttt{Non-Negative}(x) \wedge \texttt{Integer}(x)]).$$

The translation is incorrect. Give a correct translation, and describe a scenario in which your translation and their translation evaluate to different truth values.

(b) Your friend tried to translate "There is a positive number smaller than all prime numbers" and got

$$\exists x \forall y([\texttt{Positive}(x) \wedge \texttt{Prime}(y)] \rightarrow \texttt{LessThan}(x, y)).$$

The translation is incorrect. Give a correct translation, and describe a scenario in which your translation and their translation evaluate to different truth values.

(c) Translate the sentence "For every number x, there is a number y such that for every number z: x is less than y and/or z is less than y" into predicate logic.

4. Nope [12 points]

For this question, our domain of discourse is ”People”, and you may use the following predicates:

- $\texttt{Student}(x)$: x is a student
- $\texttt{Taking311}(x)$: x is taking CSE 311
- $\texttt{LearnInduction}(x)$: x will learn induction

(a) Translate the following sentence to predicate logic [3 points]:

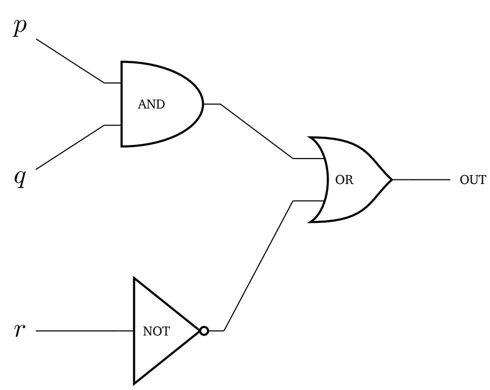
”Every student taking CSE 311 will learn induction”

(b) Negate the predicate logic sentence that you translated above in part (a); give your answer in predicate logic (i.e., symbols not in English). In your final answer, make sure that each ¬ symbol is applied only to individual predicates. For example instead of $\neg(\texttt{P}(x) \wedge \texttt{Q}(x))$, you should simplify to $(\neg\,\texttt{P}(x) \vee \neg\,\texttt{Q}(x))$. [5 points]
Note: For this part, a completely correct final answer will receive full credit, but we encourage you to show work so we can give partial credit.

(c) In part (a), the translation should contain an implication. Because of this, it is possible to use the contrapositive of that implication to write an equivalent expression. Use the contrapositive to write an equivalent expression to the one you have in part (a). Give your answer in predicate logic. Your final answer must have negations applied only to individual predicates. [4 points]
Note: As in the last part, we encourage you to show work for the sake of giving partial credit.

5. Plenty of Propositional Proposals [12 points]

(a) Translate the following circuit into propositional logic. [3 points]



(b) Fill in the truth table for the proposition found in part a. [3 points]

p	q	r	?
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

(c) Write the DNF and CNF expressions for the proposition found in part a. [6 points]

6. The Truth Will Set You Free [12 points]

Bellow is the truth table for the propositional expression $p \wedge (q \vee \neg r)$.

p	q	r	$p \wedge (q \vee \neg r)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

The DNF of it is $(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r)$. Use propositional equivalences to show that the DNF is equivalent to the original expression.
That is, show $(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \equiv p \wedge (q \vee \neg r)$. We've done the first few steps of the proof for you.

$$\begin{aligned}
(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) &\equiv (p \wedge [q \wedge r]) \vee (p \wedge [q \wedge \neg r]) \vee (p \wedge [\neg q \wedge \neg r]) && \text{Associative law 3x} \\
&\equiv p \wedge ((q \wedge r) \vee (q \wedge \neg r)) \vee (p \wedge [\neg q \wedge \neg r]) && \text{Distributivity} \\
&\equiv p \wedge ((q \wedge r) \vee (q \wedge \neg r) \vee (\neg q \wedge \neg r)) && \text{Distributivity} \\
&\equiv \ldots
\end{aligned}$$