Homework 2: Circuits, Boolean Algebra, and More Logic

Due date: Wednesday January 17 at 11:59 PM
If you work with others (and you should!), remember to follow the collaboration policy outlined in the syllabus.
In general, you are graded on your work's clarity and accuracy. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.
We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we expect. You can have longer explanations, but explanations significantly longer than necessary may receive deductions.

Be sure to read the grading guidelines on the assignments page.

1. Proof [23 points]

In Lecture 3 we gave a symbolic proof that $(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (\neg a \vee b)$. In this problem, we will give another proof.

(a) Our intuition for the proof in class was "the last two pieces of the formula correspond to vacuous truth (when a is false)." Identify a commonality in the first two pieces of the formula and describe it.
(Your description should be similar in spirit to the one from class, but it's ok if you don't have a technical term like 'vacuous truth') [4 points]

(b) Give another proof of the formula that matches the intuition from part a instead of the intuition from class. [16 points]
Read the symbolic proof guidelines on the assignments page before you start.
Hint: your proof, if it matches your intuition from (a) will be different from the one from class – at least some of the intermediate expressions will be different.

(c) In class we labeled portions of the proof in purple with high-level descriptions of what they are doing (lecture 4 slide 21, left side). Produce similar labels for your proof. Submit your answer in the form "Steps [X] to [Y]: [label]." [3 points]
Note: The goal here is to give intuition for what is happening at a higher level than individual steps.

2. Circuit du Soleil [10 points]

In this problem, we'll construct two propositions in terms of the variables p, q, r and then use these propositions to build a circuit that computes a binary function $M(p, q, r)$.

(a) Give a propositional logic formula containing only the variables p and q which evaluates to p when q is true and evaluates to false when q is false. [2 points]

(b) Give a propositional logic formula containing only the variables q and r which evaluates to true when q is true and evaluates to $\neg r$ when q is false. [2 points]

(c) Now consider the binary function $M(p, q, r)$ which is defined as:

$$M(p, 0, r) := \neg r$$

$$M(p, 1, r) := \neg p$$

Draw a circuit that takes p, q, r as input, uses only AND, OR, and NOT gates, and outputs $M(x, y, z)$. Your gates should not take more than two inputs.
Your answer for this part **must** combine your answers from (a) and (b)! [6 points]

3. May the Contrapositive be with You [15 points]

(a) In order for Baby Yoda to become a Jedi, he must be force sensitive and forgo any attachments.

   (i) convert this sentence to propositional logic (as on homework 1, ensure you're giving variables to **atomic propositions**, not compound ones). [2 points]

   (ii) take the contrapositive symbolically, and simplify so that $\neg$ signs are next to atomic propositions (i.e. only single variables). You are not required to show work for this part [2 points]

   (iii) translate the contrapositive back to English. [3 points]

   (iv) Compare English you've written down to original implication. Do they mean the same thing? (Just say "yes" or "no" here) [0.5 points]

(b) A Jedi has mastered the force only if they can wield a lightsaber.

   Repeat steps (i)-(iv) from (a) for this sentence.

4. Two of a Kind [20 points]

(a) Translate the Boolean Algebra expression $(Y \cdot X')' + X' \cdot (Y + X + Y')$ to Propositional Logic. Use the variables x and y to represent the propositions $X = 1$ and $Y = 1$, respectively. [2 points]

(b) Prove that your solution to (a) is a tautology using a chain of equivalences. [16 points]

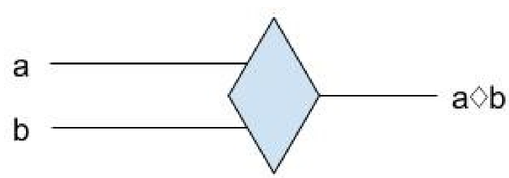
(c) Why do we know that the Boolean Algebra expression from part (a) will always evaluate to 1? Explain (1-2 sentences). [2 points]

5. Shine Bright Like a Diamond [20 points]

Consider a new logical operator $\diamond$ (*diamond* in latex). For this problem, we define $a \diamond b$ by the following truth table:

A	B	$A \diamond B$
1	1	0
1	0	1
0	1	1
0	0	1

Even though this is a new operator, we can still use it to create circuits representing other operators we have seen before! Here is an example drawing of the diamond gate:



(a) Using only $\diamond$ gates and the input a, create a circuit whose output represents $\neg a$. You may use multiple copies of a as inputs if required.

(b) Using only $\diamond$ gates and the inputs a and b, create a circuit that's output represents $a \wedge b$. You may use multiple copies of a, b as inputs if required.

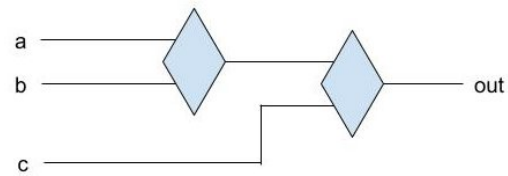
Hint You might find it helpful for this problem to start by drawing a truth table with both $a \diamond b$ and $a \wedge b$.

(c) Using only $\diamond$ gates and the inputs a and b, create a circuit that's output represents $a \vee b$. You may use multiple copies of a, b as inputs if required.

5.1. More Diamonds!

Now that you have created these circuits, notice how the truth value of $a \diamond b$ is equivalent to $\neg(a \wedge b)$ (check the truth table to confirm this fact). This fact means we can perform translations from circuits and expressions using $\diamond$ to circuits and expressions using $\neg, \wedge, \vee$.

For example, consider this circuit:



This circuit can be read as $(a \diamond b) \diamond c$, which using the fact about $\diamond$ is equivalent to $\neg(\neg(a \wedge b) \wedge c)$.

(d) Translate your circuit for $a \wedge b$ from part (b) into propositional logic notation (i.e. notation using $\neg, \wedge$). Do not simplify in this part; your final expression should only contain $\neg$ and $\wedge$ symbols and copies of a and b.

(e) As this expression represents $a \wedge b$, use a chain of logical equivalences to simplify your expression from the previous part into $a \wedge b$

6. A Tale of Two $\forall$ [12 points]

Consider the following two expressions:

$$\forall x(\mathtt{P}(x) \vee \mathtt{Q}(x)) \qquad \forall x(\mathtt{P}(x)) \vee \forall x(\mathtt{Q}(x))$$

(a) Give a domain of discourse and definitions of P and Q such that these expressions are **not** equivalent. Explain why your examples work (1-2 sentences). [6 points]

(b) Give a domain of discourse and definitions of P and Q such that these expressions **are** equivalent. Explain why your examples work (1-2 sentences). [6 points]

7. In the Real World: Logic Beyond CS [12 points]

Background

Our main goal in 311 is to prepare you to be a better computer scientist, but some of the lessons of the course are useful in "everyday life." One of the most common errors in reasoning in the "real world" is confusing an implication for its converse. That is, thinking that $p \to q$ and $q \to p$ can always be interchanged. This error can appear in a multiple ways.

One common way is a mistake called "affirming the consequent." The error is from givens $p \to q$ and q to conclude p. That conclusion is an error – from q and $q \to p$ (the converse of $p \to q$), one can conclude p. But not from q and $p \to q$.

This mistake can also appear in much longer strings of logical reasoning. For example, it is tempting to combine "If it is raining, then we won't play softball."; "If we don't play softball, then we'll get ice cream"; "we got ice cream" into the conclusion "it is raining.", but it might not be raining! (Maybe we get ice cream if we don't play softball OR if we play softball and win).

A common reason for this error is assuming that one cause is the only possible cause. Of course, this doesn't mean the conclusion is necessarily wrong—it may be that p is in fact true! But we can't guarantee it from just the reasoning given.

GRADING NOTE: These are open-ended question with the goal of seeing that 311 and logic can be seen in real life. Thus, this will be graded more leniently. We want you to focus more on finding examples of 311 in real life and having fun with this problem, rather than being stressed about if your answer is sufficient.

7.1. In Real Life [6 points]

Find an example of someone making an error involving mixing up an implication and its converse. This could be in the news (like a politician or opinion piece), in culture (like a TV clip or an advertisement), or just something your friend said. If you can't think of one, you may use one of the examples in the last section.

(a) Write down a quote of the person making the logical error.

(b) If you can link us to a source, do so here (don't worry about formatting). If you can't (because there's no record of the statement), just say that.

(c) From the quote you have in (a), define propositions p, q and translate it to propositional logic. What givens are they asserting and what is their conclusion (i.e. what do they intend to argue for)? Double check that they really are making an error.

(d) Suggest replacing one (or more) of the implications with their converse in the propositional logic you have in (c). With the change, can one reach their desired conclusion without making any logical error? (1-3 sentences)

(e) Do you think the converse(s) you inserted are true? Or at least "often true"?[1] Explain in 1-3 sentences.

Some options

You're encouraged to keep your eyes out for this error in real life! Or to think about places you might have seen it. If you cannot find one, you might choose one of these options instead:

- This clip from The Simpsons.
- This debate comment from then-presidential-candidate Mike Huckabee.
- The clip from the childrens' show "If you give a mouse a cookie" available on Ed.

7.2. At UW [6 points]

Take (and attach) a photo of some sign (preferably at UW, but we will accept any signs you see in your day-to-day life), break the contents of the sign into atomic propositions and provide the logical translation of this sign. Please also mention where you found this sign (so perhaps if any of your TA's are bored, we can do a logical sign trek :D). Example:

[1]For example, it might not be true that "If I have my umbrella, then it is raining" (since I also bring my umbrella when it snows), but snow is so rare that the implication would 'often' be true.

We can represent this with the atomic propositions :
p : One is an unattended child.
q : One will be given an espresso.
r : One will be given a free puppy.
$p \to q \wedge r$

(a) Attach a photo of some sign / iconography you have found.

(b) If you feel comfortable, mention where you found this sign.

(c) Define atomic propositions.

(d) Translate the contents of the sign into logic using the atomic propositions defined above.

8. [Extra Credit] 1111+1111 = Integer Overflow?

In this question you will construct a binary calculator equipped with the addition operator. For ease of understanding and writing, feel free to use Java syntax for loop structures and method structures.
Assume you are given two binary integers in the form of `boolean[]`s, where the first one is some `a = boolean[n]`, of length n, the second is some `b = boolean[m]`, of length m. You may assume that $m \leq n$. In this representation, every entry is a boolean (we're in binary!) $a[0]$ is the "least-significant-bit" (the "one's place").

Your goal is to return their sum in binary (return it as a `boolean[]` as well, in the same format). For this problem you may **only** use the boolean operators $\neg, \wedge, \vee$ in finding the sum of these two binary integers.

You are free to use java-like syntax (including java structures like loops) along with propositional logic notation. But you may not use any operator other than $\neg, \wedge, \vee$ to combine/alter the booleans.

(for a greater challenge, limit yourself to only two of these operators, and consider why you cannot solve this task using only one of these operators).

Hint: Consider breaking this problem down. First, consider the maximal possible size of the output array. Then, isolate a single "column" in the addition, and consider what information you need to find the corresponding output digit. Then, try adding small binary numbers by hand and seeing how you would construct these addition operations using only boolean operators. Good luck!

9. Feedback [1 point]

Answer these questions on the separate gradescope box for this question.

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment (excluding any extra credit questions, if applicable)? Report your estimate to the nearest hour.
- Which problem did you spend the most time on?
- Any other feedback for us?