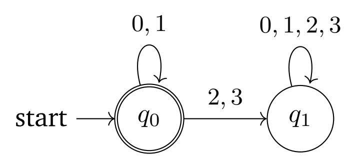
Section 08: Solutions

1. DFAs, Stage 1

Construct DFAs to recognize each of the following languages. Let $\Sigma = \{0, 1, 2, 3\}$.

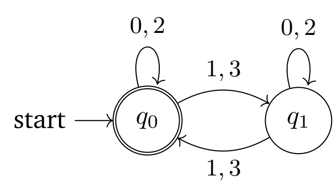
(a) All binary strings.
Solution:



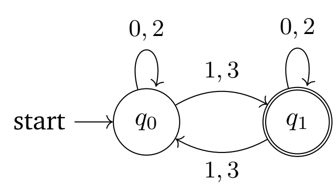
q_0: binary strings

q_1: strings that contain a character which is not 0 or 1.

(b) All strings whose digits sum to an even number.
Solution:



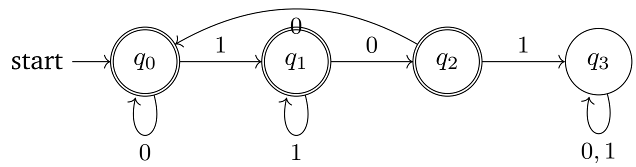
(c) All strings whose digits sum to an odd number.
Solution:



2. DFAs, Stage 2

Construct DFAs to recognize each of the following languages. Let $\Sigma = \{0, 1\}$.

(a) All strings which do not contain the substring 101.
Solution:



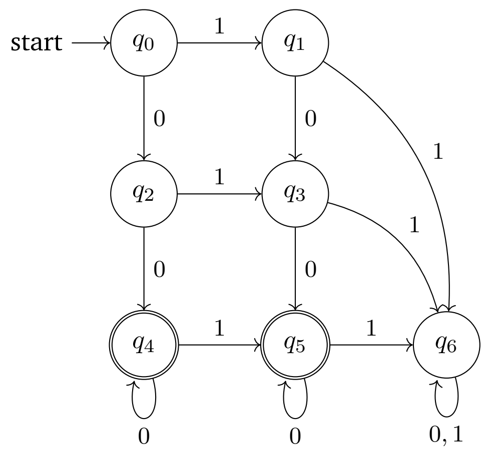
q_3: string that contain 101.

q_2: strings that don't contain 101 and end in 10.

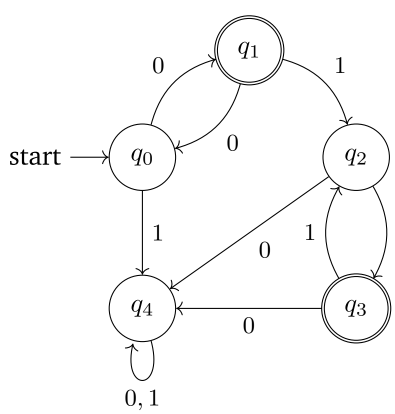
q_1: strings that don't contain 101 and end in 1.

q_0: ε, 0, strings that don't contain 101 and end in 00.

(b) All strings containing at least two 0's and at most one 1.
Solution:

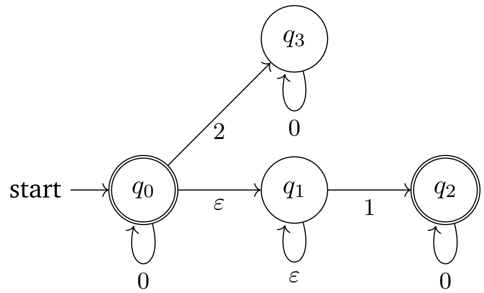


(c) All strings containing an even number of 1's and an odd number of 0's and not containing the substring 10.
Solution:



3. NFAs

(a) What language does the following NFA accept?

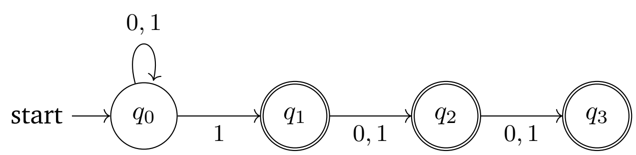


Solution:

All strings of only 0's and 1's not containing more than one 1.

(b) Create an NFA for the language "all binary strings that have a 1 as one of the last three digits".
Solution:

The following is one such NFA:



4. Irregularity

(a) Let $\Sigma = \{0, 1\}$. Prove that $\{0^n 1^n 0^n \ : \ n \geq 0\}$ is not regular.
Solution:

Let $L = \{0^n 1^n 0^n \ : \ n \geq 0\}$. Let D be an arbitrary DFA, and suppose for contradiction that D accepts L. Consider $S = \{0^n 1^n : n \geq 0\}$. Since S contains infinitely many strings and D has a finite number of states, two strings in S must end up in the same state. Say these strings are $0^i 1^i$ and $0^j 1^j$ for some $i, j \geq 0$ such that $i \neq j$. Append the string 0^i to both of these strings. The two resulting strings are:

$a = 0^i 1^i 0^i$ Note that $a \in L$.

$b = 0^j 1^j 0^i$ Note that $b \notin L$, since $i \neq j$.

Since a and b end up in the same state, but $a \in L$ and $b \notin L$, that state must be both an accept and reject state, which is a contradiction. Since D was arbitrary, there is no DFA that recognizes L, so L is not regular.

(b) Let $\Sigma = \{0, 1, 2\}$. Prove that $\{0^n (12)^m \ : \ n \geq m \geq 0\}$ is not regular.
Solution:

Let $L = \{0^n (12)^m \ : \ n \geq m \geq 0\}$. Let D be an arbitrary DFA, and suppose for contradiction that D accepts L. Consider $S = \{0^n : n \geq 0\}$. Since S contains infinitely many strings and D has a finite number of states, two strings in S must end up in the same state. Say these strings are 0^i and 0^j for some $i, j \geq 0$ such that $i > j$. Append the string $(12)^i$ to both of these strings. The two resulting strings are:

$a = 0^i (12)^i$ Note that $a \in L$.

$b = 0^j (12)^i$ Note that $b \notin L$, since $i > j$.

Since a and b end up in the same state, but $a \in L$ and $b \notin L$, that state must be both an accept and reject state, which is a contradiction. Since D was arbitrary, there is no DFA that recognizes L, so L is not regular.

5. Countability

Prove that the set $S := \{3x : x \in \mathbb{N}\}$ is countable.

Solution:

Define the function $f : \mathbb{N} \to S$ as follows:

$$f(0) = 0, f(1) = 3, f(2) = 6, f(3) = 9, \cdots, f(n) = 3n.$$

f is injective, since by definition for every element $s \in S$, there is some $x \in \mathbb{N}$ such that $s = 3x$, and $f(x) = s$, so every element of s is reached by f.

6. Countability 2

You are a pirate. You begin in a square on a 2D grid which is infinite in all directions. In other words, wherever you are, you may move up, down, left, or right. Some single square on the infinite grid has treasure on it. Find a way to ensure you find the treasure in finitely many moves.

How can you use this idea allow you to prove that integer pairs on the 2D plane (ie. the set $\{(x, y) : x, y \in \mathbb{R}\}$) are countabley infinite? Do you see how to extend the argument to prove the rationals are countable?

Solution:

Explore the square you are currently on. Explore the unexplored perimeter of the explored region until you find the treasure (your path will look a bit like a spiral).

This path also provides a injection from the natural integers to integer points on the 2D plane.

We can write any rational number r as a quotient of integers, $r = a/b$. Then define a mapping from integer points on the 2D plane to the rationals as $(x, y) \to x/y$. Clearly every rational number will be reached, so this mapping is an injection. By combining it with the injection above we get an injection from the natural numbers to the rationals, proving the rational numbers are countable.

7. Uncountability

Prove that $\mathcal{P}(\mathbb{N})$ is uncountable.

Solution:

Assume for the sake of contradiction that $\mathcal{P}(\mathbb{N})$ is countable. This means we can define an enumeration of elements S_i in $\mathcal{P}(\mathbb{N})$. Let s_i be the binary set representation of S_i in $\mathcal{P}(\mathbb{N})$. For example, for the set 0, 1, 2, the binary set representation would be 111000 . . . We then construct a new subset $X \subset \mathbb{N}$ such that $x[i] = s_i[i]$ (that is, $x[i]$ is 1 if $si[i]$ is 0, and $x[i]$ is 0 otherwise). Note that X is not any of S_i, since it differs from S_i on the ith natural number. However, X still represents a valid subset of the natural numbers, which means our enumeration is incomplete, which is a contradiction. Since the above proof works for any listing of $\mathcal{P}(\mathbb{N})$, no listing can be created for $\mathcal{P}(\mathbb{N})$, and therefore $\mathcal{P}(\mathbb{N})$ is uncountable.