Section 09: Final Review

1. Irregular Languages

(a) Let $\Sigma = \{0, 1\}$. Prove that $\{0^n 1^n 0^n \ : n \geq 0\}$ is irregular.

(b) Let $\Sigma = \{0, 1, 2\}$. Prove that $\{0^n (12)^m : n \geq m \geq 0\}$ is not regular.

2. Countability

(a) Prove that the set $\{5x : x \in \mathbb{N}\}$ is countable.

(b) Prove that the set of irrational numbers is uncountable. You may use the fact that the real numbers are uncountable and the rationals are countable. **Hint:** Every real number is either rational or irrational.

(c) Prove that $\mathcal{P}(\mathbb{N})$ is uncountable.

3. Translations

Translate the following sentences into logical notation if the English statement is given or to an English statement if the logical statement is given, taking into account the domain restriction.

Let the domain of discourse be students and courses.

Use predicates Student, Course, CseCourse to do the domain restriction.

You can use $\mathsf{Taking}(x, y)$ which is true if and only if x is taking y. You can also use $\mathsf{JacobTeaches}(x)$ if and only if Jacob teaches x and $\mathsf{ContainsTheory}(x)$ if and only if x contains theory.

(a) Every student is taking some course.

(b) There is a student that is not taking every CSE course.

(c) Some student has taken only one CSE course.

(d) $\forall x[(\mathsf{Course}(x) \land \mathsf{JacobTeaches}(x)) \to \mathsf{ContainsTheory}(x)]$

(e) $\exists x(\mathsf{CseCourse}(x) \land \mathsf{JacobTeaches}(x) \land \mathsf{ContainsTheory}(x) \land \forall y((\mathsf{CseCourse}(y) \land \mathsf{JacobTeaches}(y)) \to x = y)$

4. Induction

(a) A Husky Tree is a tree built by the following definition:

Basis: A single gold node is a Husky Tree.

Recursive Rules:

(i) Let T_1 and T_2 be two Husky Trees, both with root nodes colored gold. Make a new purple root node and attach the roots of T_1 and T_2 to the new node to make a new Husky Tree.

(ii) Let T_1 and T_2 be two Husky Trees, both with root nodes colored purple. Make a new purple root node and attach the roots of T_1 and T_2 to the new node to make a new Husky Tree.

(iii) Let T_1 and T_2 be two Husky Trees, one with a purple root and the other with a gold root. Make a new gold root node and attach the roots of T_1 and T_2 to the new node to make a new Husky Tree.

Use structural induction to show that for every Husky Tree:

- If it has a purple root, then it has an even number of leaves.
- If it has a gold root, then it has an odd number of leaves.

(b) Use induction to prove that for every positive integer n,

$$1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1).$$

5. Languages

(a) Construct a regular expression that represents binary strings where no occurence of 11 is followed by a 0.

(b) Construct a CFG that represents the following language: $\{1^x 2^y 3^y 4^x : x, y \geq 0\}$.

(c) Construct a DFA that recognizes the language of all binary strings which,when interpreted as a binary number, are divisible by 3. e.g. 11 is 3 in base-10, so should be accepted while 111 is 7 in base-10, so should be rejected. The first bit processed will be the most-significant bit. Hint: you need to keep track of the remainder mod 3. What happens to a binary number when you add a 0 at the end? A 1? It's a lot like a shift operation...