Section 06: Structural Induction, Strong Induction, and Sets

1. Reversing a Binary Tree

Recall the following recursive definition of the set of Trees from lecture:
Basis Step: $\mathsf{null} \in \mathsf{Tree}$

Recursive Step: If $L, R \in \mathsf{Tree}$ and $a \in \mathbb{Z}$, then $(L, a, R) \in \mathsf{Tree}$.

Now consider the following recursive definitions of the functions sum and $\mathsf{reverse}$:
$\mathsf{sum}(\mathsf{null}) = 0$
$\mathsf{sum}((L, a, R)) = a + \mathsf{sum}(L) + \mathsf{sum}(R)$

$\mathsf{reverse}(\mathsf{null}) = \mathsf{null}$
$\mathsf{reverse}((L, a, R)) = (\mathsf{reverse}(R), a, \mathsf{reverse}(L))$

Prove that for every Tree $T \in \mathsf{Tree}$ that $\mathsf{sum}(\mathsf{reverse}(T)) = \mathsf{sum}(T)$

2. Treeshake

We define simple binary trees as the recursive set $\mathcal{B}$:

Basis Step: $\bullet \in \mathcal{B}$.
Recursive Step: If $L, R \in \mathcal{B}$, then $(L, \bullet, R) \in \mathcal{B}$.

Note that these are slightly different than the trees defined in class. These trees cannot be null.

Define the following functions on simple binary trees:

$$\mathtt{edges}(t) = \begin{cases} 0 & \text{if } t = \bullet \\ 2 + \mathtt{edges}(L) + \mathtt{edges}(R) & \text{if } t = (L, \bullet, R) \end{cases}$$

$$\mathtt{degree}(t) = \begin{cases} 1 & \text{if } t = \bullet \\ 3 & \text{if } t = (L, \bullet, R) \end{cases}$$

$$\mathtt{sum}(t) = \begin{cases} \mathtt{degree}(t) & \text{if } t = \bullet \\ \mathtt{degree}(t) + \mathtt{sum}(L) + \mathtt{sum}(R) & \text{if } t = (L, \bullet, R) \end{cases}$$

Prove that for all $t \in \mathcal{B}$, $\mathtt{sum}(t) = 2 \cdot \mathtt{edges}(t) + 1$.

This is a special case of an important result in graph theory called the *Handshaking Lemma*. You will probably use it a lot if you end up taking an algorithms or graph theory course.

3. A Set Theory Interlude

(a) Prove or disprove: For all sets A, B, C if $A \cap C = B \cap C$ then $A = B$.

(b) Prove or disprove: For all sets A, B, C if $A \cup C = B \cup C$ then $A = B$.

(c) Prove or disprove: For all sets A, B, C if $A \cup C = B \cup C$ and $A \cap C = B \cap C$ then $A = B$.

4. Geometric Sum

Suppose that a and r are real numbers with $r \neq 1$. Prove by induction that for all $n \in \mathbb{N}$:

$$a + ar + ar^2 + ... + ar^n = \frac{a \cdot r^{n+1} - a}{r - 1}$$