Section 06: Structural Induction, Strong Induction, and Sets

1. Reversing a Binary Tree

Recall the following recursive definition of the set of Trees from lecture:

Basis Step: null ∈ Tree

Recursive Step: If L, R ∈ Tree and a ∈ Z, then (L, a, R) ∈ Tree.

Now consider the following recursive definitions of the functions sum and reverse:

\[ \text{sum}(\text{null}) = 0 \]
\[ \text{sum}((L, a, R)) = a + \text{sum}(L) + \text{sum}(R) \]

\[ \text{reverse}(\text{null}) = \text{null} \]
\[ \text{reverse}((L, a, R)) = (\text{reverse}(R), a, \text{reverse}(L)) \]

Prove that for every Tree \( T \in \text{Tree} \) that \( \text{sum} (\text{reverse} (T)) = \text{sum} (T) \)

2. Treeshake

We define simple binary trees as the recursive set \( B \):

Basis Step: \( \bullet \in B \).

Recursive Step: If \( L, R \in B \), then \( (L, \bullet, R) \in B \).

Note that these are slightly different than the trees defined in class. These trees cannot be null.

Define the following functions on simple binary trees:

\[ \text{edges}(t) = \begin{cases} 0 & \text{if } t = \bullet \\ 2 + \text{edges}(L) + \text{edges}(R) & \text{if } t = (L, \bullet, R) \end{cases} \]

\[ \text{degree}(t) = \begin{cases} 1 & \text{if } t = \bullet \\ 3 & \text{if } t = (L, \bullet, R) \end{cases} \]

\[ \text{sum}(t) = \begin{cases} \text{degree}(t) & \text{if } t = \bullet \\ \text{degree}(t) + \text{sum}(L) + \text{sum}(R) & \text{if } t = (L, \bullet, R) \end{cases} \]

Prove that for all \( t \in B \), \( \text{sum}(t) = 2 \cdot \text{edges}(t) + 1 \).

This is a special case of an important result in graph theory called the Handshaking Lemma. You will probably use it a lot if you end up taking an algorithms or graph theory course.

3. A Set Theory Interlude

(a) Prove or disprove: For all sets \( A, B, C \) if \( A \cap C = B \cap C \) then \( A = B \).

(b) Prove or disprove: For all sets \( A, B, C \) if \( A \cup C = B \cup C \) then \( A = B \).
(c) Prove or disprove: For all sets $A, B, C$ if $A \cup C = B \cup C$ and $A \cap C = B \cap C$ then $A = B$.

4. Geometric Sum

Suppose that $a$ and $r$ are real numbers with $r \neq 1$. Prove by induction that for all $n \in \mathbb{N}$:

$$a + ar + ar^2 + \ldots + ar^n = \frac{a \cdot r^{n+1} - a}{r - 1}$$