Section 06: Solutions

1. Reversing a Binary Tree

Recall the following recursive definition of the set of Trees from lecture:
Basis Step: $\mathsf{null} \in \mathsf{Tree}$

Recursive Step: If $L, R \in \mathsf{Tree}$ and $a \in \mathbb{Z}$, then $(L, a, R) \in \mathsf{Tree}$.

Now consider the following recursive definitions of the functions sum and reverse:
$\mathsf{sum}(\mathsf{null}) = 0$
$\mathsf{sum}((L, a, R)) = a + \mathsf{sum}(L) + \mathsf{sum}(R)$

$\mathsf{reverse}(\mathsf{null}) = \mathsf{null}$
$\mathsf{reverse}((L, a, R)) = (\mathsf{reverse}(R), a, \mathsf{reverse}(L))$

Prove that for every Tree $T \in \mathsf{Tree}$ that $\mathsf{sum}(\mathsf{reverse}(T)) = \mathsf{sum}(T)$

Solution:

Let $P(T)$ be "$\mathsf{sum}(\mathsf{reverse}(T)) = \mathsf{sum}(T)$". We show $P(T)$ for all $T \in \mathsf{Tree}$ by structural induction.

Base Case. We show P(null) holds. Observe that the LHS evaluates to $\mathsf{sum}(\mathsf{reverse}(\mathsf{null})) = \mathsf{sum}(\mathsf{null}) = 0$. Observe that the RHS evaluates to $\mathsf{sum}(\mathsf{null}) = 0$. Since $0 = 0$, the base case holds.

Induction Hypothesis. Suppose $\mathsf{P}(L)$ and $\mathsf{P}(R)$ hold for some arbitrary trees $L, R \in \mathsf{Tree}$. That is, $\mathsf{sum}(\mathsf{reverse}(L)) = \mathsf{sum}(L)$ and $\mathsf{sum}(\mathsf{reverse}(R)) = \mathsf{sum}(R)$

Induction Step. Goal: $\mathsf{sum}(\mathsf{reverse}((L, a, R))) = \mathsf{sum}((L, a, R))$ for any $a \in \mathbb{Z}$

Let $a \in \mathbb{Z}$ be arbitrary. Then observe that:

$$\begin{aligned}
\mathsf{sum}(\mathsf{reverse}((L, a, R))) &= \mathsf{sum}((\mathsf{reverse}(R), a, \mathsf{reverse}(L))) && \text{[By Definition of reverse]} \\
&= a + \mathsf{sum}(\mathsf{reverse}(R)) + \mathsf{sum}(\mathsf{reverse}(L)) && \text{[By Definition of sum]} \\
&= a + \mathsf{sum}(R) + \mathsf{sum}(L) && \text{[By IH]} \\
&= \mathsf{sum}((L, a, R)) && \text{[By Definition of sum]}
\end{aligned}$$

This proves P((L, a, R)).

Conclusion. Thus, P(T) holds for all trees $T \in \mathsf{Tree}$ by structural induction.

2. Treeshake

We define simple binary trees as the recursive set $\mathcal{B}$:

Basis Step: $\bullet \in \mathcal{B}$.
Recursive Step: If $L, R \in \mathcal{B}$, then $(L, \bullet, R) \in \mathcal{B}$.

Note that these are slightly different than the trees defined in class. These trees cannot be null.

Define the following functions on simple binary trees:

$$\texttt{edges}(t) = \begin{cases} 0 & \text{if } t = \bullet \\ 2 + \texttt{edges}(L) + \texttt{edges}(R) & \text{if } t = (L, \bullet, R) \end{cases}$$

$$\texttt{degree}(t) = \begin{cases} 1 & \text{if } t = \bullet \\ 3 & \text{if } t = (L, \bullet, R) \end{cases}$$

$$\texttt{sum}(t) = \begin{cases} \texttt{degree}(t) & \text{if } t = \bullet \\ \texttt{degree}(t) + \texttt{sum}(L) + \texttt{sum}(R) & \text{if } t = (L, \bullet, R) \end{cases}$$

Prove that for all $t \in \mathcal{B}$, $\texttt{sum}(t) = 2 \cdot \texttt{edges}(t) + 1$.

This is a special case of an important result in graph theory called the *Handshaking Lemma*. You will probably use it a lot if you end up taking an algorithms or graph theory course.

Solution:

Let $P(t) :=$"$\texttt{sum}(t) = 2 \cdot \texttt{edges}(t) + 1$". We will prove $P(t)$ holds for all $t \in \mathcal{B}$ by structural induction.

Basis Step: From the definitions,

$$\texttt{sum}(\bullet) = \texttt{degree}(\bullet) = 1$$

and

$$2 \cdot \texttt{edges}(\bullet) + 1 = 2(0) + 1 = 1$$

Since both are 1, the base case is satisfied.

Inductive Hypothesis: Suppose $P(L)$ and $P(R)$ for some arbitrary simple binary trees L, R.

Inductive Step: Let $t = (L, \bullet, R)$. We will show $P(t)$.

$$\begin{aligned} \texttt{sum}(t) &= \texttt{degree}(t) + \texttt{sum}(L) + \texttt{sum}(R) \\ &= 3 + \texttt{sum}(L) + \texttt{sum}(R) \\ &= 3 + 2 \cdot \texttt{edges}(L) + 1 + 2 \cdot \texttt{edges}(R) + 1 && \text{by IH} \\ &= 5 + 2 \cdot \texttt{edges}(L) + 2 \cdot \texttt{edges}(R) \\ &= 1 + 2(2) + 2 \cdot \texttt{edges}(L) + 2 \cdot \texttt{edges}(R) \\ &= 1 + 2(2 + \texttt{edges}(L) + \texttt{edges}(R)) \\ &= 1 + 2 \cdot \texttt{edges}(t) \end{aligned}$$

This proves $P(t)$.

Conclusion: Therefore, $P(t)$ holds for all $t \in \mathcal{B}$ by structural induction.

3. A Set Theory Interlude

(a) Prove or disprove: For all sets A, B, C if $A \cap C = B \cap C$ then $A = B$.

Solution:

This claim is false. Consider $A = \{1\}$, $B = \{2\}$, and $C = \varnothing$. Then $A \cap C = \varnothing = B \cap C$, but $A \neq B$.

(b) Prove or disprove: For all sets A, B, C if $A \cup C = B \cup C$ then $A = B$.

Solution:

This claim is false. Consider $A = \{1\}$, $B = \{2\}$, and $C = \{1, 2\}$. Then $A \cup C = \{1, 2\} = B \cup C$, but $A \neq B$.

(c) Prove or disprove: For all sets A, B, C if $A \cup C = B \cup C$ and $A \cap C = B \cap C$ then $A = B$.

Solution:

This claim is true. Let sets A, B, C be arbitrary, and suppose that $A \cup C = B \cup C$ and $A \cap C = B \cap C$. We prove by two subset proofs.

$\subseteq$: We aim to show that $A \subseteq B$. Let $x \in A$ be arbitrary.
Case 1: $x \in A$ and $x \in C$. Then by definition of intersection, $x \in A \cap C$. Since $A \cap C = B \cap C$, $x \in B \cap C$. Then by definition of intersection, $x \in B$ and $x \in C$. So $x \in B$. Since x was arbitrary, $A \subseteq B$.
Case 2: $x \in A$ and $x \notin C$. Since $x \in A$, by definition of union, $x \in A \cup C$. Since $A \cup C = B \cup C$, $x \in B \cup C$. Then by definition of union, $x \in B$ or $x \in C$. But since $x \notin C$, we have $x \in B$. Since x was arbitrary, $A \subseteq B$.
Thus in all cases $A \subseteq B$.

$\supseteq$: Now we aim to show that $B \subseteq A$. This argument follows similarly to the previous, since the setup is symmetric.

Thus we have shown that $A \subseteq B$ and $B \subseteq A$, so $A = B$, as desired. Since A, B, C were arbitrary, the claim holds.

4. Geometric Sum

Suppose that a and r are real numbers with $r \neq 1$. Prove by induction that for all $n \in \mathbb{N}$:

$$a + ar + ar^2 + ... + ar^n = \frac{a \cdot r^{n+1} - a}{r - 1}$$

Solution:

Let P(n) be "$a + ar + ar^2 + ... + ar^n = \frac{a \cdot r^{n+1} - a}{r-1}$". We will prove P($n$) for all $n \in \mathbb{N}$ by induction.

Base Case. The LHS simplifies to $a \cdot r^0 = a$. The RHS simplifies to $\frac{a \cdot r^{0+1} - a}{r-1} = \frac{ar - a}{r-1} = \frac{a(r-1)}{r-1} = a$. Since $a = a$, the base case holds.

Inductive Hypothesis. Assume that P(k) holds true for some arbitrary $k \geq 0$. Then $a + ar + ar^2 + ... + ar^k = \frac{a \cdot r^{k+1} - a}{r-1}$.

Inductive Step Goal: Show $a + ar + ar^2 + ... + ar^{k+1} = \dfrac{a \cdot r^{k+2} - a}{r - 1}$

$$
\begin{aligned}
a + ar + ar^2 + ... + ar^{k+1} &= a + ar + ar^2 + ... + ar^k + ar^{k+1} && \text{[Show another term inside "..."]} \\
&= \frac{ar^{k+1} - a}{r - 1} + ar^{k+1} && \text{[Inductive Hypothesis]} \\
&= \frac{ar^{k+1} - a}{r - 1} + \frac{ar^{k+1}(r - 1)}{r - 1} && \text{[Finding Common Denominator]} \\
&= \frac{ar^{k+1} - a + ar^{k+1}r - ar^{k+1}}{r - 1} && \text{[Algebra]} \\
&= \frac{ar^{k+2} - a}{r - 1} && \text{[Algebra]}
\end{aligned}
$$

Therefore $\mathrm{P}(k + 1)$ holds.

Conclusion. $\mathrm{P}(n)$ holds for all $n \in \mathbb{N}$ by induction.