Section 05: Induction and Midterm Review

1. Bernoulli's Inequality

Prove that for every real number x and even integer r,

$$(1+x)^r \geq 1 + rx.$$

Hint: Use the definition of even to write $r = 2k$ and then induct on k.

2. Donald Duck's Devious Duel

Donald duck challenges you to a game. The rules are simple; there are a bunch of cookies on the table, you and Donald will take turns eating either 1, 2 or 3 cookies. Whoever eats the last cookie wins. You will go first. Prove that no matter how many cookies there are on the table, if the number of cookies on the table is divisible by 4 then there is a way for Donald Duck to always win.

3. Cantelli's Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function f:

$$\begin{aligned}
&f(0) = 0\\
&f(1) = 1\\
&f(n) = 2f(n-1) - f(n-2) \text{ for } n \geq 2
\end{aligned}$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n. That is, construct a formula for $f(n)$ and prove its correctness.

4. A Horse of a Different Color

Did you know that all dogs are named Dubs? It's true. Maybe. Let's prove it by induction. The key is talking about groups of dogs, where every dog has the same name.

Let $P(i)$ mean "all groups of i dogs have the same name." We prove $\forall n\ P(n)$ by induction on n.

Base Case: $P(1)$ Take an arbitrary group of one dog, all dogs in that group all have the same name (there's only the one, so it has the same name as itself).

Inductive Hypothesis: Suppose $P(k)$ holds for some arbitrary k.

Inductive Step: Consider an arbitrary group of $k+1$ dogs. Arbitrarily select a dog, D, and remove it from the group. What remains is a group of k dogs. By inductive hypothesis, all k of those dogs have the same name. Add D back to the group, and remove some other dog D'. We have a (different) group of k dogs, so the inductive hypothesis applies again, and every dog in that group also shares the same name. All $k+1$ dogs appeared in at least one of the two groups, and our groups overlapped, so all of our $k+1$ dogs have the same name, as required.

Conclusion: We conclude $P(n)$ holds for all n by the principle of induction.

Recalling that Dubs is a dog, we have that every dog must have the same name as him, so every dog is named Dubs.

This proof cannot be correct (the proposed claim is false). Where is the bug?

5. Number Theory

If $n^2 - 6n + 5$ is even then n is odd.

Hint: Use contrapositive (notice how much easier the algebra becomes!).