Section 05: Solutions

1. Bernoulli's Inequality

Prove that for every real number x and even integer r,

$$(1+x)^r \geq 1+rx.$$

Hint: Use the definition of even to write $r = 2k$ and then induct on k.

Solution:

Since r is even, we can write $r = 2k$ for some integer $k \geq 0$. We will prove this by induction on k.

Define $P(k) :=$ "$(1+x)^{2k} \geq 1+2kx$" and prove using induction that $P(k)$ holds for all integers $k \geq 0$.

Base Case. If $k = 0$ then,

$$(1+x)^{2k} = (1+x)^0 = 1 \geq 1+(0)x = 1+2kx.$$

So the claim holds for $k = 0$.

Inductive Hypothesis. Suppose that $P(k)$ holds for some arbitrary nonnegative integer k.

Inductive Step. We wish to show $P(k+1)$.

$$\begin{aligned}
(1+x)^{2(k+1)} &= (1+x)^{2k}(1+x)^2 & \\
&\geq (1+2kx)(1+x)^2 & \text{Inductive Hypothesis} \\
&= (1+2kx)(1+2x+x^2) & \\
&\geq (1+2kx)(1+2x) & \text{since } x^2 \geq 0 \\
&= 1+2x+2kx+4kx^2 & \\
&\geq 1+2x+2kx & \text{since } x^2 \geq 0 \\
&= 1+(2(k+1))x. &
\end{aligned}$$

Which proves $P(k+1)$, note we used the inductive hypothesis, the fact $P(k)$ is true on the second line.

Thus $P(k)$ holds for all non-negative integers k by the principle of induction.

2. Donald Duck's Devious Duel

Donald duck challenges you to a game. The rules are simple; there are a bunch of cookies on the table, you and Donald will take turns eating either 1, 2 or 3 cookies. Whoever eats the last cookie wins. You will go first. Prove that no matter how many cookies there are on the table, if the number of cookies on the table is divisible by 4 then there is a way for Donald Duck to always win.

Solution:

Since the number of cookies is divisible by 4, we may write it as the number $4n$, where n is an integer at least 1.

Define $P(n) :=$"If the table has $4n$ cookies and it is your turn then Donald Duck can win." We will prove $P(n)$ holds for all non-negative integers by induction.

Base Case. When $n = 1$, there are 4 cookies and on the first turn you must eat at least 1 but not more than 3 cookies. So there are between 1 and 3 remaining cookies, which means Donald can eat them all on his turn and win.

Inductive Hypothesis. Suppose that $P(n)$ holds for an arbitrary integer $n \geq 1$.

Inductive Step. We wish to show $P(n+1)$, so suppose there are $4(n+1) = 4n+4$ cookies. Let $c \in \{1,2,3\}$ be the number of cookies you eat on your turn. Then on his turn, Donald should eat $4-c$ many cookies (note $4-c \in \{1,2,3\}$). If he does this, then there will be $4n+4-c-(4-c) = 4n$ many cookies remaining and it will be your turn again, so we can apply the inductive hypothesis and we are done.

Thus we have shown $P(n)$ holds by the principle of induction.

3. Cantelli's Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function f:

$$
\begin{aligned}
&f(0) = 0 \\
&f(1) = 1 \\
&f(n) = 2f(n-1) - f(n-2) \text{ for } n \geq 2
\end{aligned}
$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n. That is, construct a formula for $f(n)$ and prove its correctness.

Solution:

Let $P(n)$ be "$f(n) = n$". We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by strong induction on n.

Base Cases $(n = 0, n = 1)$**:** $f(0) = 0$ and $f(1) = 1$ by definition.

Inductive Hypothesis: Assume that $P(0) \wedge P(1) \wedge \ldots P(k)$ hold for some arbitrary $k \geq 1$.

Inductive Step: We show $P(k+1)$:

$$
\begin{aligned}
f(k+1) &= 2f(k) - f(k-1) && \text{[Definition of } f\text{]} \\
&= 2(k) - (k-1) && \text{[Induction Hypothesis]} \\
&= k+1 && \text{[Algebra]}
\end{aligned}
$$

Conclusion: $P(n)$ is true for all $n \in \mathbb{N}$ by principle of strong induction.

4. A Horse of a Different Color

Did you know that all dogs are named Dubs? It's true. Maybe. Let's prove it by induction. The key is talking about groups of dogs, where every dog has the same name.

Let $P(i)$ mean "all groups of i dogs have the same name." We prove $\forall n\ P(n)$ by induction on n.

Base Case: $P(1)$ Take an arbitrary group of one dog, all dogs in that group all have the same name (there's only the one, so it has the same name as itself).

Inductive Hypothesis: Suppose $P(k)$ holds for some arbitrary k.

Inductive Step: Consider an arbitrary group of $k+1$ dogs. Arbitrarily select a dog, D, and remove it from the group. What remains is a group of k dogs. By inductive hypothesis, all k of those dogs have the same name. Add D back to the group, and remove some other dog D'. We have a (different) group of k dogs, so the inductive hypothesis applies again, and every dog in that group also shares the same name. All $k+1$ dogs appeared in at least one of the two groups, and our groups overlapped, so all of our $k+1$ dogs have the same name, as required.

Conclusion: We conclude $P(n)$ holds for all n by the principle of induction.

Recalling that Dubs is a dog, we have that every dog must have the same name as him, so every dog is named Dubs.

This proof cannot be correct (the proposed claim is false). Where is the bug?

Solution:

The bug is in the final sentence of the inductive step. We claimed that the groups overlapped, i.e. that some dog was in both of them. That's true for large k, but not when $k+1=2$. When $k=2$, D is in a group by itself, and D' was in a group by itself. The inductive hypothesis holds (D has the only name in its subgroup, and D' has the only name in its subgroup) but returning to the full group $\{D, D'\}$ we cannot conclude that they share a name.

From there everything unravels. $P(1) \not\to P(2)$, so we cannot use the principle of induction. It turns out this is the **only** bug in the proof. The argument in the inductive step is correct as long as $k+1 > 2$. But that implication is always vacuous, since $P(2)$ is false.

5. Number Theory

If $n^2 - 6n + 5$ is even then n is odd.

Hint: Use contrapositive (notice how much easier the algebra becomes!). **Solution:**

We prove by contrapositive. Suppose that n is not odd, so it is even and there is some integer k so that $n = 2k$. We want to show that $n^2 - 6n + 5$ is odd. To this end,

$$\begin{aligned} n^2 - 6n + 5 &= (2k)^2 - 6(2k) + 5 \\ &= 4k^2 - 12k + 5 \\ &= 2(2k^2 - 6k + 2) + 1. \end{aligned}$$

Since $2k^2 - 6k + 2$ is an integer, we have shown $n^2 - 6 + 5$ is odd, which proves the contrapositive.