Section 02: Domain Restriction and Translation

1. Quantifier Translation

For each part, appropriately define predicates and propositions and translate each of the sentences into logic, using quantifiers and domain restriction as necessary.

(a) **Solution:**

Define the following predicates:

$$\begin{aligned} \text{TestCase}(x) &:= \text{“}x \text{ is a test-case.”} \\ \text{Bug}(x) &:= \text{“}x \text{ is a bug.”} \\ \text{IsValid}(x) &:= \text{“}x \text{ is valid.”} \\ \text{Catches}(x, y) &:= \text{“}x \text{ catches } y\text{.”} \\ \text{Fails}(x) &:= \text{“}x \text{ fails.”} \end{aligned}$$

(i) Each test-case catches at least one bug.

Solution:

$$\forall x \exists y (\text{TestCase}(x) \to (\text{Bug}(y) \land \text{Catches}(x, y))).$$

(ii) If all the test-cases are valid, then there is a bug as long as there is a test-case which fails.

Solution:

$$(\forall x (\text{TestCase}(x) \to \text{IsValid}(x))) \to ((\exists y (\text{TestCase}(y) \land \text{Fails}(y)) \to \exists z\, \text{Bug}(z))).$$

(iii) There are bugs which are not caught by any test-cases.

Solution:

$$\exists x (\text{Bug}(x) \land \forall y (\text{TestCase}(y) \to \neg \text{Catches}(y, x))).$$

(b) **Solution:**

Define the following predicates:

$$\begin{aligned} \text{Server}(x) &:= \text{“}x \text{ is a server.”} \\ \text{IsDown}(x) &:= \text{“}x \text{ is down.”} \\ \text{InDiagonosticMode}(x) &:= x \text{ is in diagnostic mode.} \end{aligned}$$

and the propositions:

$$\begin{aligned} \text{LoadtimeSlow} &:= \text{“The load-time on my laptop is} > 10\text{ms.”} \\ \text{LoadtimeInf} &:= \text{“The load-time on my laptop is infinite.”} \\ \text{HaveInternet} &:= \text{“I have an internet connection.”} \end{aligned}$$

(i) At least one server is down if the website load-time on my laptop is >10ms and there is no server in diagnostic mode.

Solution:

$$(\text{LoadtimeSlow} \land \neg\exists y(\text{Server}(y) \land \text{IsInDiagnosticMode}(y))) \to (\exists x(\text{Server}(x) \land \text{IsDown}(x))).$$

(ii) When the website load-time on my laptop is infinite either all of the servers are down or I do not have an internet connection.

Solution:

$$\text{LoadtimeInf} \to (\forall x(\text{Server}(x) \to \text{IsDown}(x)) \lor \neg\text{HaveInternet}).$$

2. Order of Quantifiers and Domain Restriction

(a) (i) Define some predicates and translate the following into logic, using quantifiers: Every UW student has a favorite class. *Be sure to do domain restriction!*

Solution:

Define the propositions:

$$\begin{aligned} \text{Student}(x) &:= \text{“}x \text{ is a UW student.”} \\ \text{Class}(x) &:= \text{“}x \text{ is a class.”} \\ \text{Favorite}(x,y) &:= \text{“}y \text{ is a favorite of } x.\text{”} \end{aligned}$$

And the translation is,

$$\forall x \exists y(\text{Student}(x) \to (\text{Class}(y) \land \text{Favorite}(x,y)).$$

(ii) Exchange the order of quantifiers in English and then translate this new sentence to logic. *You will now have to perform the other type of domain restriction!*

Solution:

The new sentence is,

There is a favorite class of all UW students.

Which in logic becomes,

$$\exists y \forall x(\text{Class}(y) \land (\text{Student}(x) \to \text{Favorite}(x,y)).$$

(iii) Notice that if you naively exchange the order of quantifiers in the logic ($\forall a \exists b$ becomes $\exists b \forall a$) from your answer to part (i), you get something slightly different from (ii). It turns out these logical claims are almost equivalent but not quite. Can you prove they are not equivalent? **Hint:** Design a domain of discourse where one evaluates to false and the other to true.

Solution:

The two statements are,

$$\begin{aligned} &(1)\ \exists y \forall x(\text{Class}(y) \land (\text{Student}(x) \to \text{Favorite}(x,y)) \\ &(2)\ \forall x \exists y(\text{Student}(x) \to (\text{Class}(y) \land \text{Favorite}(x,y)). \end{aligned}$$

If we take the domain of discourse to be anything which doesn't contain either students or classes (say something silly like farm animals), then Class$(y) \equiv$ F and so (1) will always be false. On the other hand, since also Student$(x) \equiv$ F, this means (2) will be vacuously true.

3. Equivalence

Prove that the following are equivalent using a formal proof:

$$\neg p \to (q \to r) \equiv q \to (p \vee r).$$

Solution:

Proof.

$$\begin{aligned}
\neg p \to (q \to r) &\equiv \neg\neg p \vee (q \to r) && \text{[Law of Implication]}\\
&\equiv p \vee (q \to r) && \text{[Double Negation]}\\
&\equiv p \vee (\neg q \vee r) && \text{[Law of Implication]}\\
&\equiv (\neg q \vee r) \vee p && \text{[Commutativity]}\\
&\equiv \neg q \vee (r \vee p) && \text{[Associativity]}\\
&\equiv q \to (r \vee p) && \text{[Law of Implication]}
\end{aligned}$$

□

4. Formal Proof

Write a formal proof that shows that if $p \vee q$, $q \rightarrow r$ and $r \rightarrow s$ all hold, then so must $\neg p \rightarrow s$.

Solution:

Proof.

1. $p \vee q$	[Given]
2. $q \rightarrow r$	[Given]
3. $r \rightarrow s$	[Given]
4.1. $\neg p$	[Assumption]
4.2. q	[Elim $\vee$: 1, 4.1]
4.3. r	[Modus Ponens:] 4.2, 2
4.4. s	[Modus Ponens:] 4.3, 3
4. $\neg p \rightarrow s$	[Direct Proof Rule]

□

5. Quantifier Switch

Consider the following pairs of sentences. For each pair, determine if one implies the other, if they are equivalent, or neither.

(a) $\forall x\, \forall y\; P(x, y)$ $\qquad$ $\forall y\, \forall x\; P(x, y)$ **Solution:**

These sentences are the same; switching universal quantifiers makes no difference.

(b) $\exists x\, \exists y\; P(x, y)$ $\qquad$ $\exists y\, \exists x\; P(x, y)$ **Solution:**

These sentences are the same; switching existential quantifiers makes no difference.

(c) $\forall x\ \exists y\ P(x, y)$ $\forall y\ \exists x\ P(x, y)$ **Solution:**

These are only the same if P is symmetric (i.e., the order of the arguments doesn't matter). If the order of the arguments does matter, then these are different statements. For instance, if $P(x, y)$ is "$x < y$", then the first statement says "for every x, there is a corresponding y such that $x < y$", whereas the second says "for every y, there is a corresponding x such that $x < y$". In other words, in the first statement y is a function of x, and in the second x is a function of y.

If your domain of discourse is "positive integers", for example, the first is true and the second is false; but for "negative integers" the second is true while the first is false.

(d) $\forall x\ \exists y\ P(x, y)$ $\exists x\ \forall y\ P(x, y)$ **Solution:**

These two statements are usually different.

(e) $\forall x\ \exists y\ P(x, y)$ $\exists y\ \forall x\ P(x, y)$ **Solution:**

The second statement is "stronger" than the first (that is, the second implies the first). For the first, y is allowed to depend on x. For the second, one specific y must work for all x. Thus if the second is true, whatever value of y makes it true, can also be plugged in for y in the first statement for every x. On the other hand, if the first statement is true, it might be that different y's work for the different x's and no single value of y exists to make the latter true.

As an example, let you domain of discourse be positive real numbers, and let $P(x, y)$ be $xy = 1$. The first statement is true (always take y to be $1/x$, which is another positive real number). The second statement is not true; it asks for a single number that always makes the product 1.