Final Exam Review
Translations

- All dogs are cute
- There is only one dog who is the best dog
Translations

- All dogs are cute
  \[ \forall x(\text{dog}(x) \rightarrow \text{cute}(x)) \]

- There is only one dog who is the best dog
  \[ \exists x(\text{dog}(x) \land \text{best}(x) \land \forall y((\text{dog}(y) \land \text{best}(y)) \rightarrow x = y)) \]
Set Theory

• Prove that \((A \cap B) \cup (A \setminus C) \subseteq A\).
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• Prove that \((A \cap B) \cup (A \setminus C) \subseteq A\).

Let \(x \in (A \cap B) \cup (A \setminus C)\) be arbitrary. By definition of union, \(x \in (A \cap B)\) or \(x \in (A \setminus C)\). We will use cases:

1. If \(x \in (A \cap B)\), then \(x \in A\) by definition of intersection.
2. If \(x \in (A \setminus C)\), then \(x \in A\) and \(x \notin C\), so clearly \(x \in A\).

In both cases \(x \in A\), thus if \(x \in (A \cap B) \cup (A \setminus C)\), \(x \in A\).

Since \(x\) was arbitrary, \((A \cap B) \cup (A \setminus C) \subseteq A\)
Number Theory

Let $p$ be a prime number $> 3$. Show that $p \equiv_6 1$ or $p \equiv_6 5$
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Let $p$ be a prime number $> 3$. Show that $p \equiv_6 1$ or $p \equiv_6 5$

Let $p$ be arbitrary. We know that by the division theorem, $p = 6q + r$ for some $q$ and $r$. This means that $p \equiv_6 r$. Consider the following cases:

$p \equiv_6 0$: This means that $6|p$, making $p$ not prime

$p \equiv_6 2$: This means that $p$ is even, and since $p > 3$, it must be not prime

$p \equiv_6 3$: This means $3|p$, making $p$ not prime

$p \equiv_6 4$: This means that $p$ is even and not prime

Therefore, $r = 1$ or $5$, so $p \equiv_6 1$ or $p \equiv_6 5$

Since $p$ was arbitrary, the claim holds
Structural Induction

We define the set \textbf{Trees} as follows:

\textbf{Basis Elements:} for any $x \in \mathbb{Z}$, $\text{Leaf}(x) \in \text{Trees}$.

\textbf{Recursive Step:} for any $x \in \mathbb{Z}$, if $L, R \in \text{Trees}$, then $\text{Branch}(x, L, R) \in \text{Trees}$.

This defines a set of \textit{binary trees} just like those defined in lecture except that we now include data in the nodes.

The following functions count the number of each type of node in the tree:

\[
\begin{align*}
\text{leaves}(\text{Leaf}(x)) & = 1 & \text{for any } x \in \mathbb{Z} \\
\text{leaves}(\text{Branch}(x, L, R)) & = \text{leaves}(L) + \text{leaves}(R) & \text{for any } x \in \mathbb{Z} \text{ and } L, R \in \text{Trees} \\
\text{branches}(\text{Leaf}(x)) & = 0 & \text{for any } x \in \mathbb{Z} \\
\text{branches}(\text{Branch}(x, L, R)) & = 1 + \text{branches}(L) + \text{branches}(R) & \text{for any } x \in \mathbb{Z} \text{ and } L, R \in \text{Trees}
\end{align*}
\]

Prove that, for any $T \in \text{Trees}$, we have $\text{leaves}(T) = \text{branches}(T) + 1$. 
Let $P(t)$ mean that “leaves(t) = branches(t) + 1”. We will prove this by structural induction for all $t \in T$.

Base Case: Leaf(x). Leaves(leaf(x)) = 1, and branches(leaf(x)) + 1 = 1, so $P(\text{leaf}(x))$ holds.

Inductive Hypothesis: Suppose $P(L)$, $P(R)$ for arbitrary $L, R \in T$.

Inductive Step: let $x$ be an arbitrary integer

Leaves(branch(x, L, R)) = Leaves(L) + Leaves(R)

= Branches(L) + Branches(R)+1 + 1 (by IH)

= Branches(Branch(x, L, R)) + 1

Which shows $P(\text{branch}(x, L, R))$

Therefore we have shown $P(t)$ for all $t \in T$ by structural induction.
Regex

- Binary Strings with an even number of 1s

- Binary strings with an odd number of 1s
Regex

- Binary Strings with an even number of 1s
  $$(0*10^*10^*)^*0^*$$

- Binary strings with an odd number of 1s
  $$(0*10^*10^*)^*0^*(0*10^*10^*)^*0^*$$
DFA, NFA

- Binary Strings where every occurrence of a 1 is followed by 00

- NFA: Binary Strings with at least 3 1s and ends with 000
DFA, NFA

- Binary Strings where every occurrence of a 1 is followed by 00

- NFA: Binary Strings with at least 3 1s and ends with 000
NFA to DFA

Convert the NFA below, which we get by applying the construction described in class to the regular expression \((\varepsilon \cup 0)(1 \cup 00)^*\):
NFA to DFA
Irregularity

- Prove that the language $1^a01^a$ is irregular
Irregularity

- Prove that the language $L = 1^a01^a$ is irregular

Suppose there is some DFA called $M$ that accepts $L$. Consider $S = \{1^a0\}$. Since there are finitely many states in $M$ and infinitely many strings in $S$, we know there exists strings $0^a1$ and $0^b1$ that go to the same state in $M$ where $a \neq b$. Now consider appending $0^a$ onto both strings. Since $0^a1$ and $0^b1$ end in the same state, $0^a10^a$ and $0^a10^b$ go to the same state let’s call $q$. $q$ needs to be accepting since $0^a10^a \in L$, but it needs to also be rejecting since $0^a10^b \notin L$ since $a \neq b$. This is a contradiction.

Thus no DFA recognizes $L$