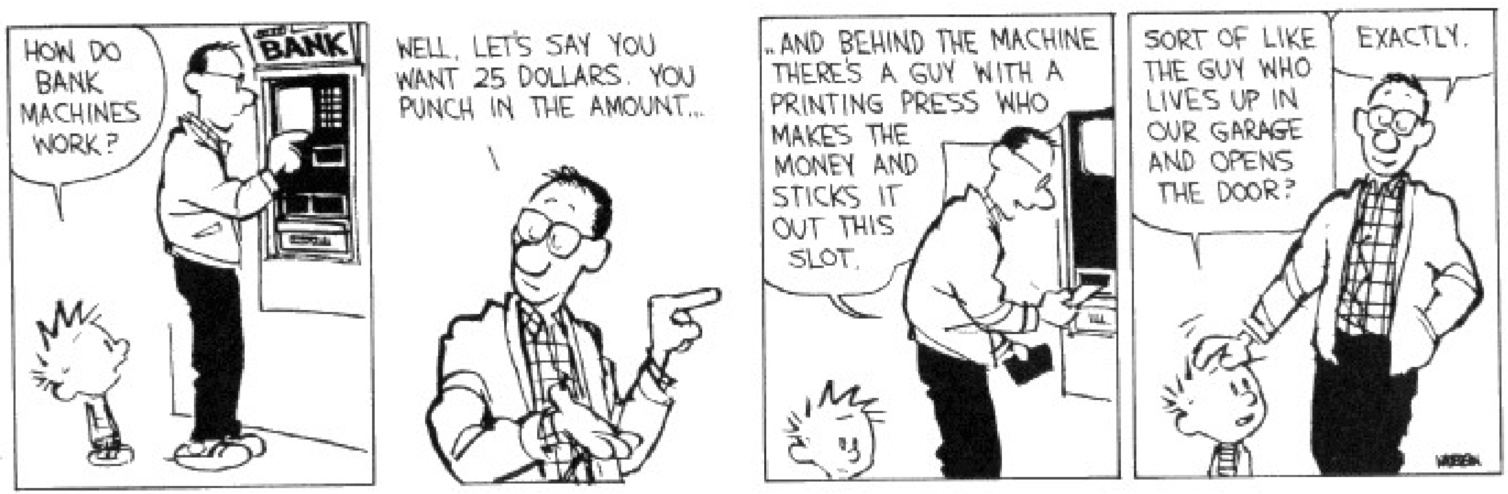
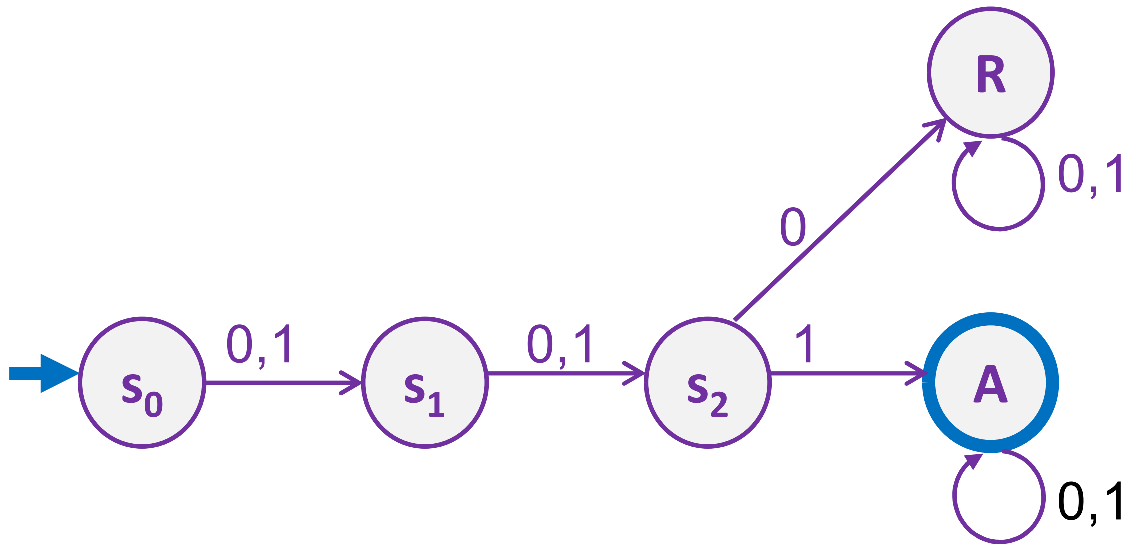
CSE 311: Foundations of Computing

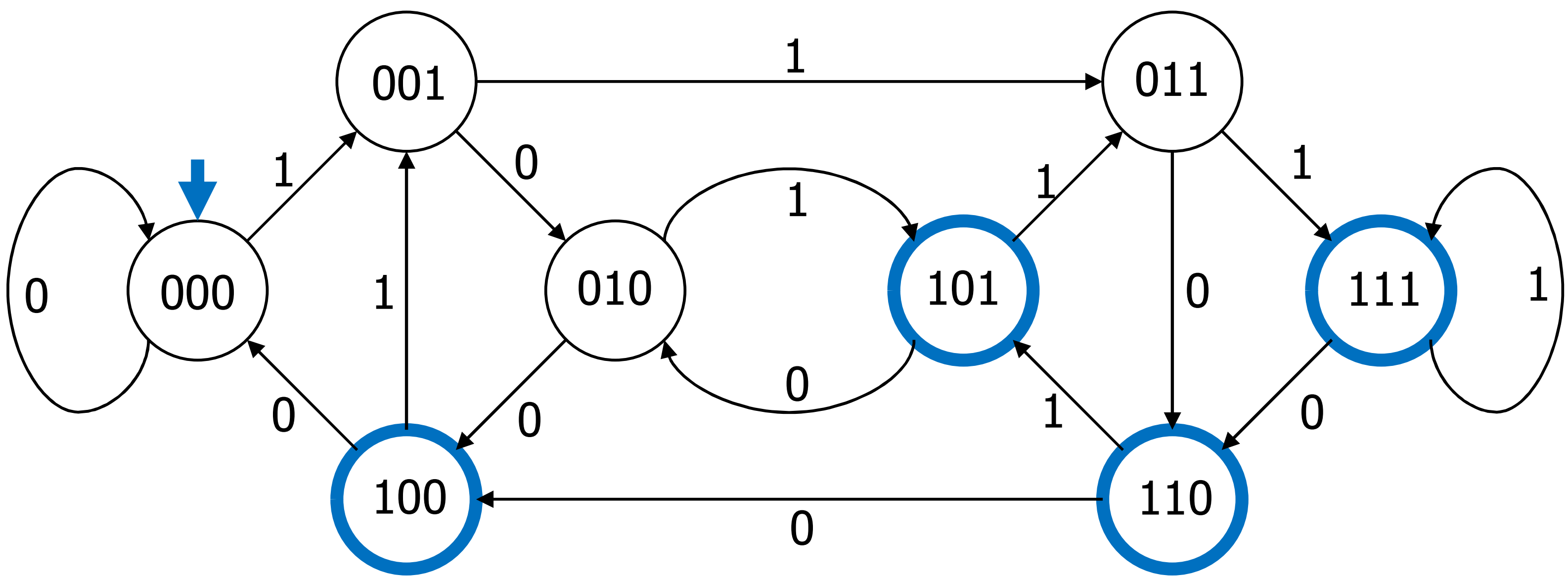
Topic 10: Finite State Machines



The set of binary strings with a 1 in the 3rd position from the start



The set of binary strings with a 1 in the 3rd position from the end



Adding Output to Finite State Machines

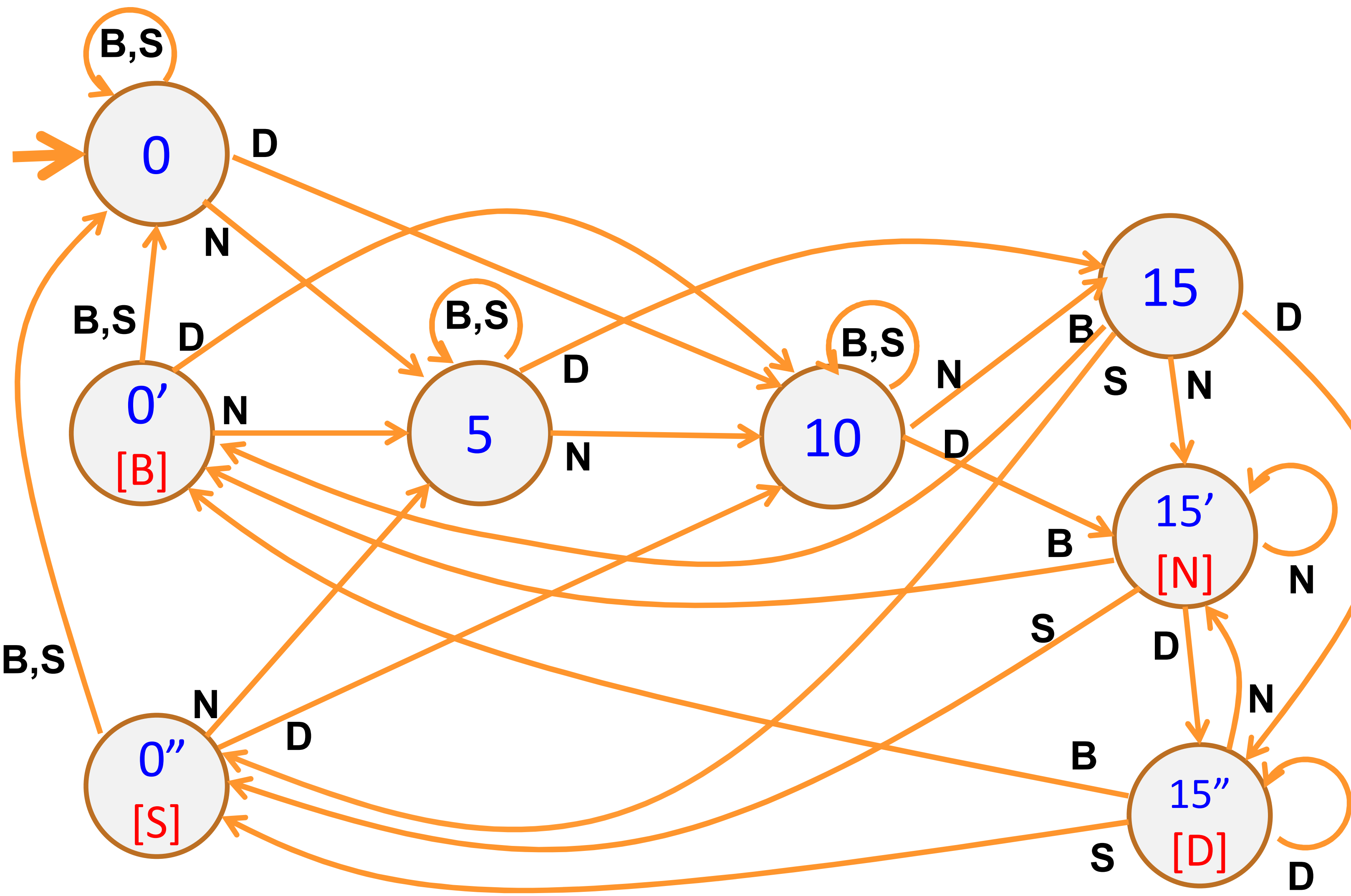
- So far we have considered finite state machines that just accept/reject strings
  – called "Deterministic Finite Automata" or DFAs

- Now we consider finite state machines *with output*
  – These are the kinds used as controllers

Vending Machine

Enter 15 cents in dimes or nickels
Press S or B for a candy bar

Vending Machine, v1.0

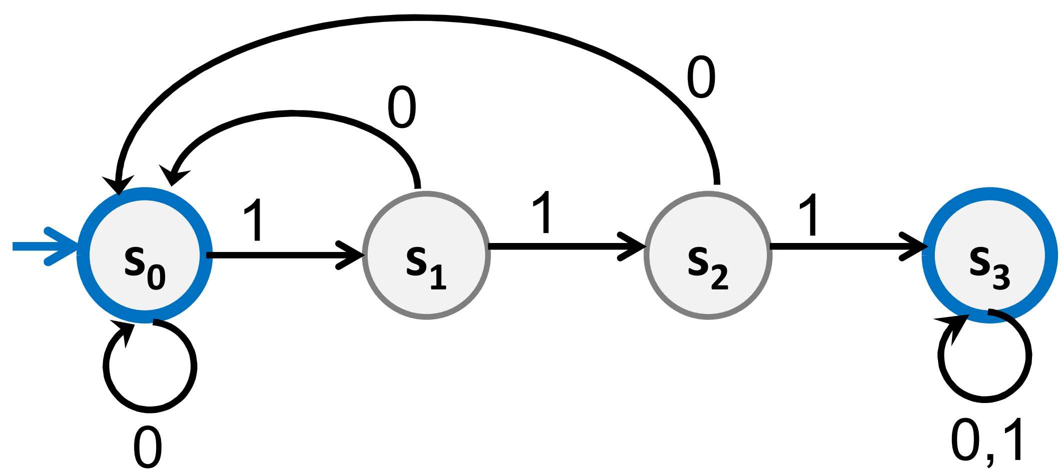


Adding additional "unexpected" transitions to cover all symbols for each state

Recall: Finite State Machines

- States
- Transitions on input symbols
- Start state and final states
- The “language recognized” by the machine is the set of strings that reach a final state from the start

Old State	0	1
s_0	s_0	s_1
s_1	s_0	s_2
s_2	s_0	s_3
s_3	s_3	s_3

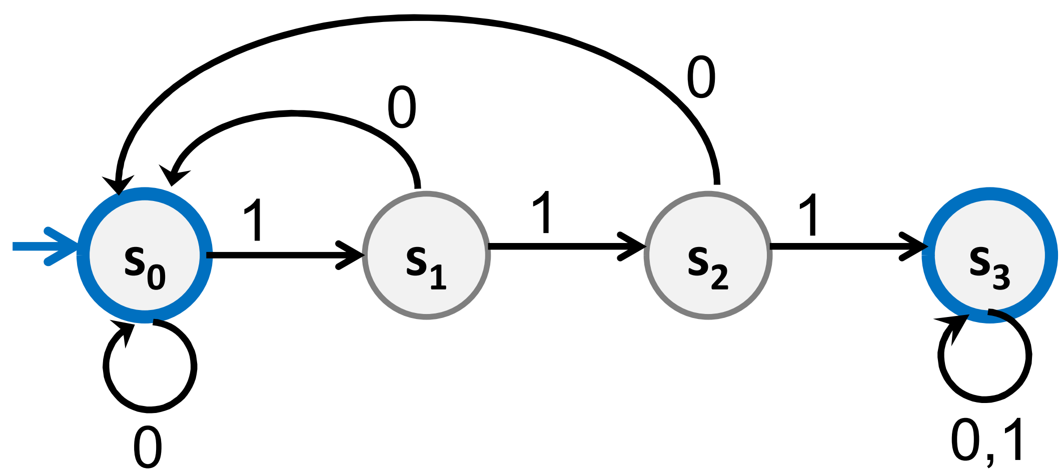


Recall: Finite State Machines

- Each machine designed for strings over some fixed alphabet Σ.

- Must have a transition defined from each state for *every* symbol in Σ.

Old State	0	1
s_0	s_0	s_1
s_1	s_0	s_2
s_2	s_0	s_3
s_3	s_3	s_3



State Minimization

- Many FSMs (DFAs) for the same problem
- Take a given FSM and try to reduce its state set by combining states
  - Algorithm will always produce the unique minimal equivalent machine (up to renaming of states) but we won't prove this

State Minimization Algorithm

- Put states into groups
- Try to find groups that can be collapsed into one state
  - states can keep track of information that isn't necessary to determine whether to accept or reject
- Group states together until we can *prove* that collapsing them can change the accept/reject result
  - find a specific string x such that:

    starting from state A, following edges according to x ends in accept

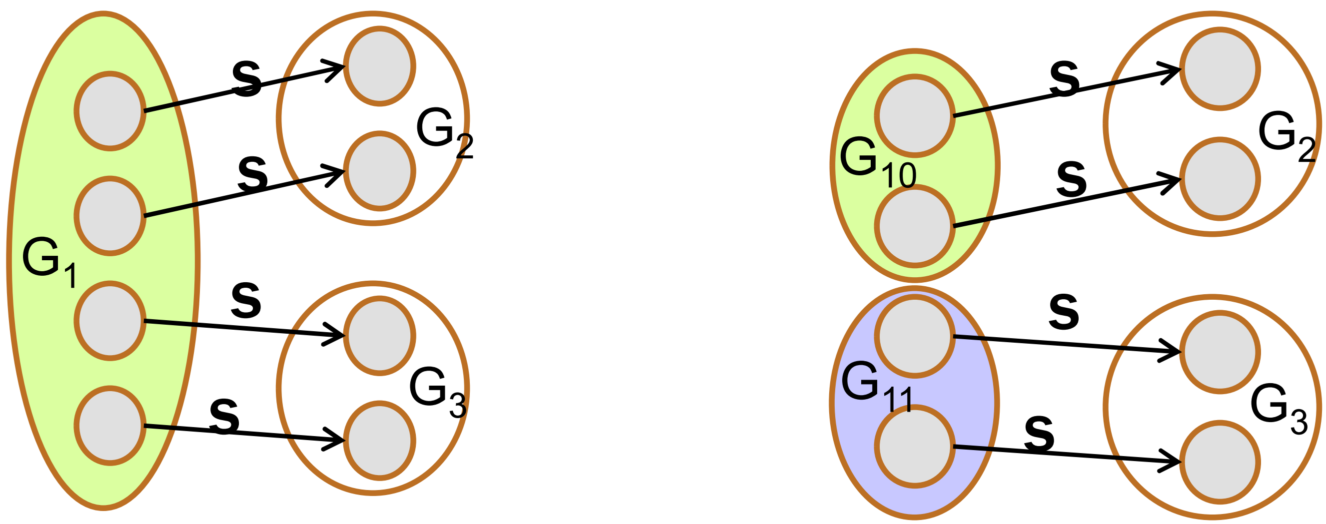
    starting from state B, following edges according to x ends in reject
  - (algorithm below could be modified to show these strings)

State Minimization Algorithm

1. Put states into groups based on their outputs (whether they accept or reject)

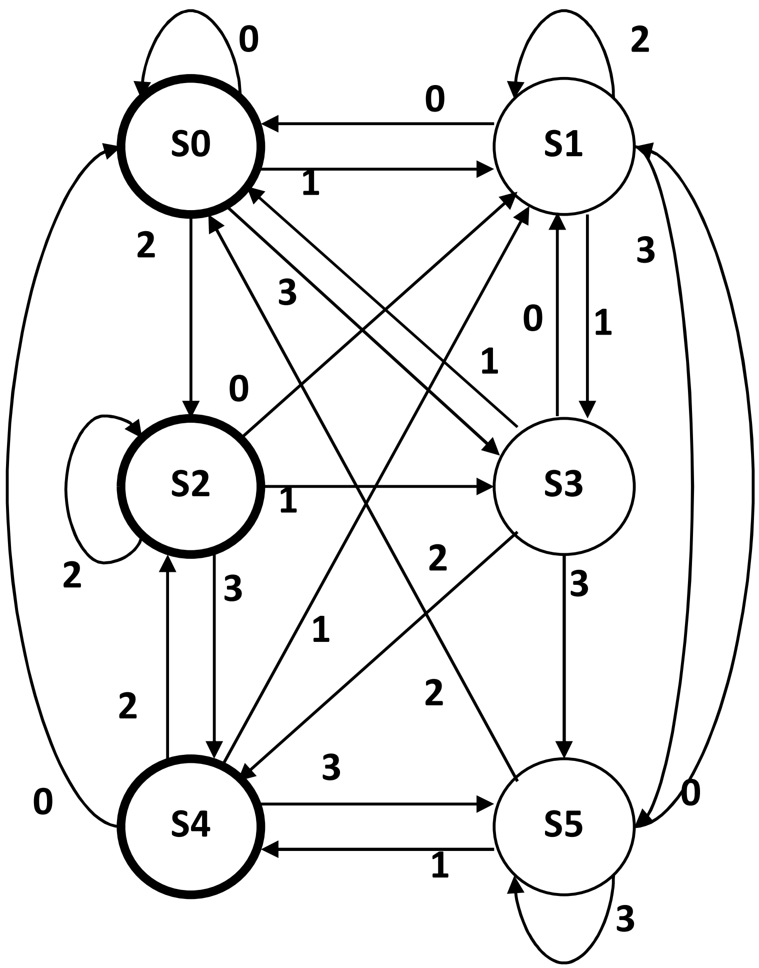
State Minimization Algorithm

1. Put states into groups based on their outputs (whether they accept or reject)
2. Repeat the following until no change happens
   a. If there is a symbol s so that not all states in a group G agree on which group s leads to, split G into smaller groups based on which group the states go to on s



3. Finally, convert groups to states

State Minimization Example

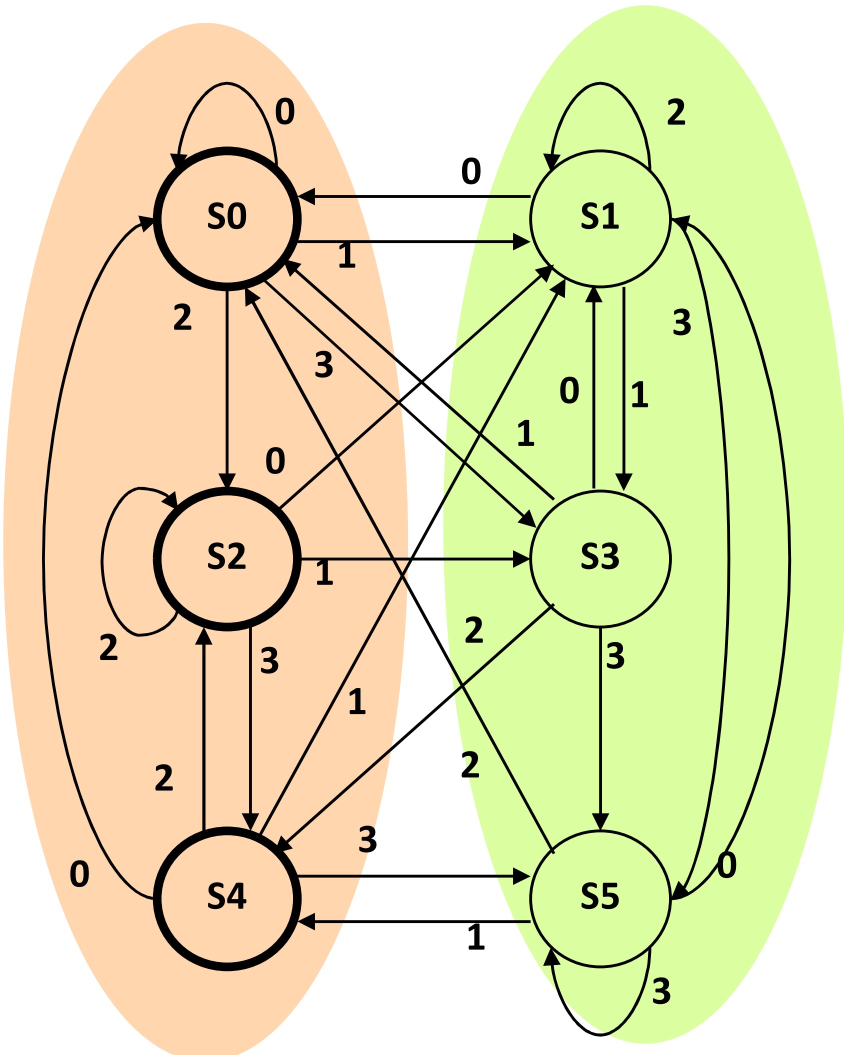


present state	next state 0	1	2	3	output
S0	S0	S1	S2	S3	1
S1	S0	S3	S1	S5	0
S2	S1	S3	S2	S4	1
S3	S1	S0	S4	S5	0
S4	S0	S1	S2	S5	1
S5	S1	S4	S0	S5	0

state
transition table

Put states into groups based on their outputs (or whether they accept or reject)

State Minimization Example



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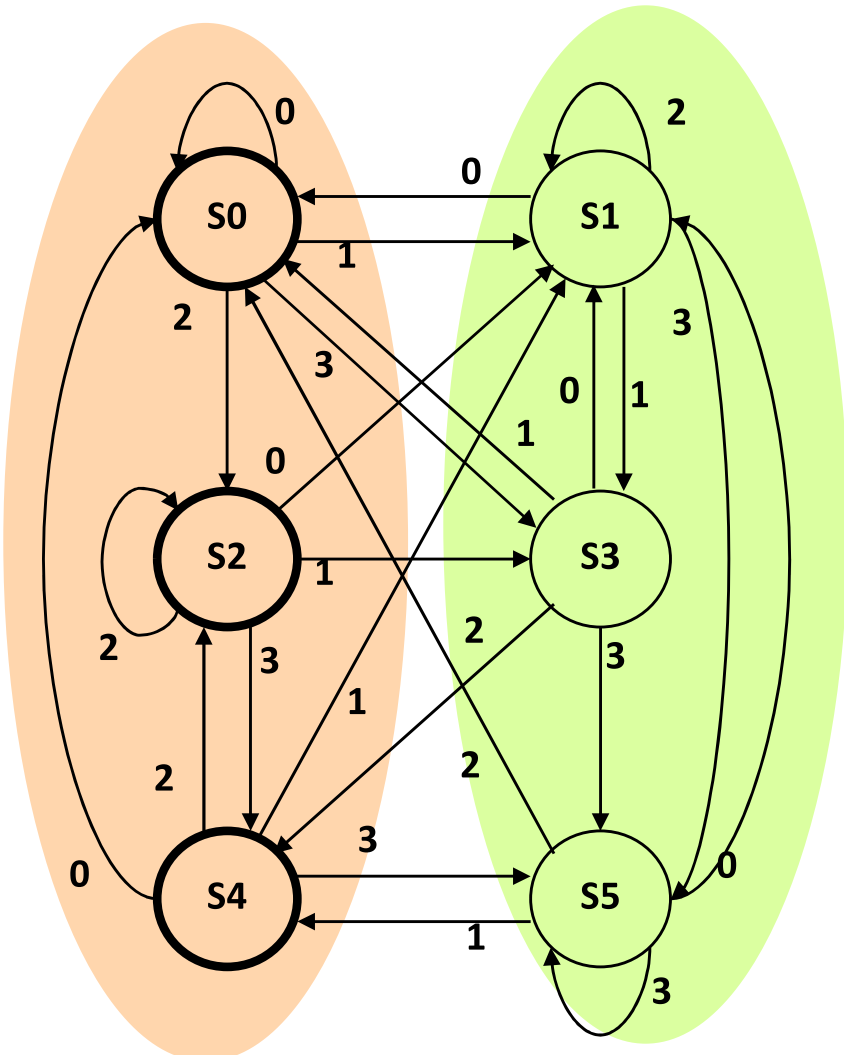
S4	S0	S1	S2	S5	1
S5	S1	S4	S0	S5	0

state
transition table

Put states into groups based on their outputs (or whether they accept or reject)

State Minimization Example



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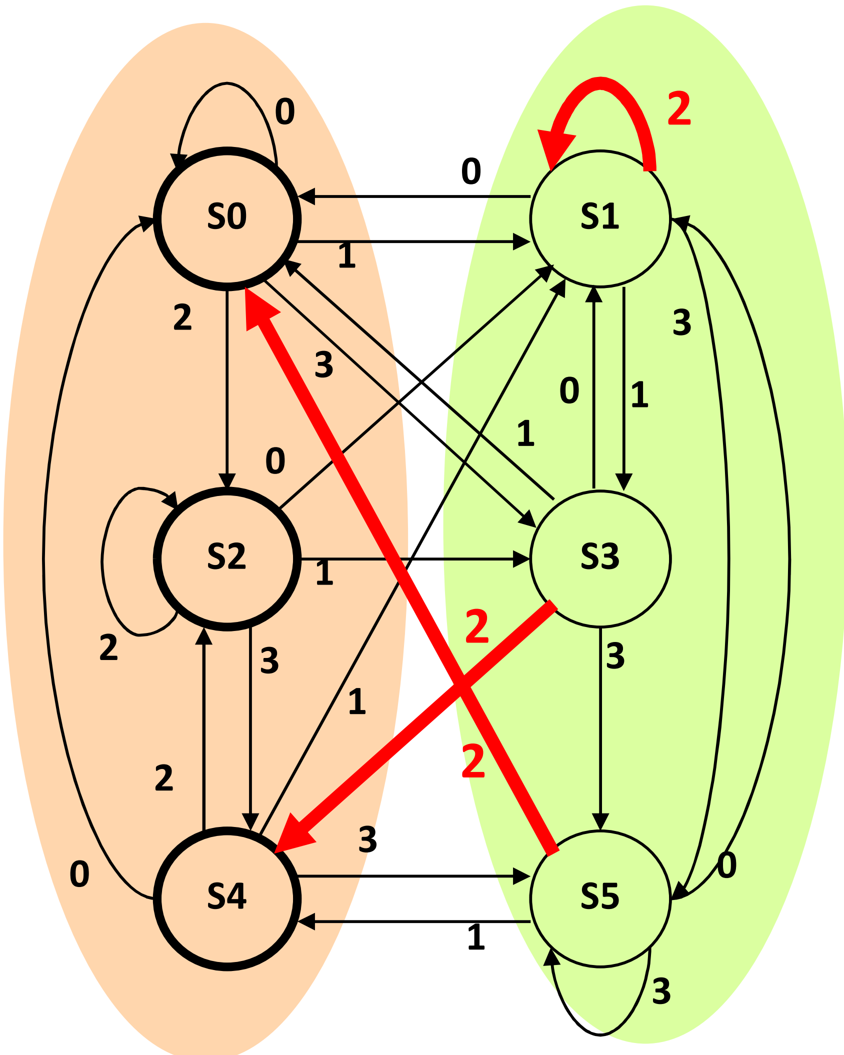
S5	S1	S4	S0	S5	0

state transition table

Put states into groups based on their outputs (or whether they accept or reject)

If there is a symbol **s** so that not all states in a group **G** agree on which group **s** leads to, split **G** based on which group the states go to on **s**

State Minimization Example



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state transition table

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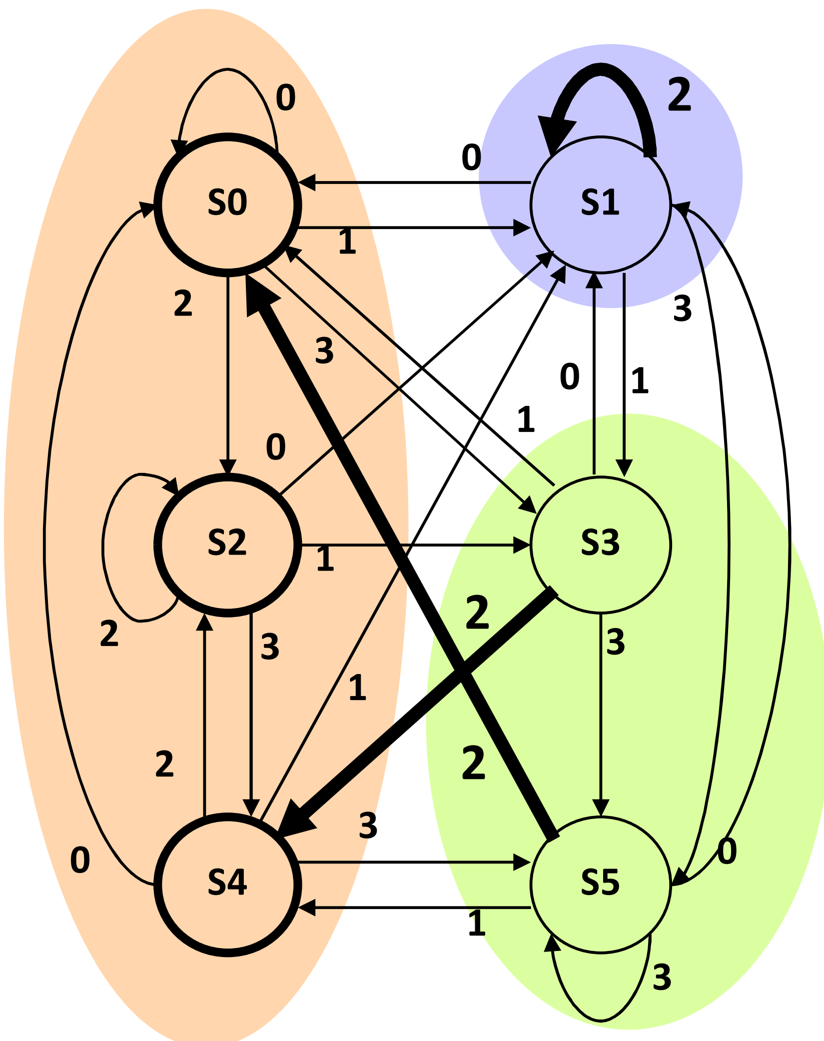

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state
transition table

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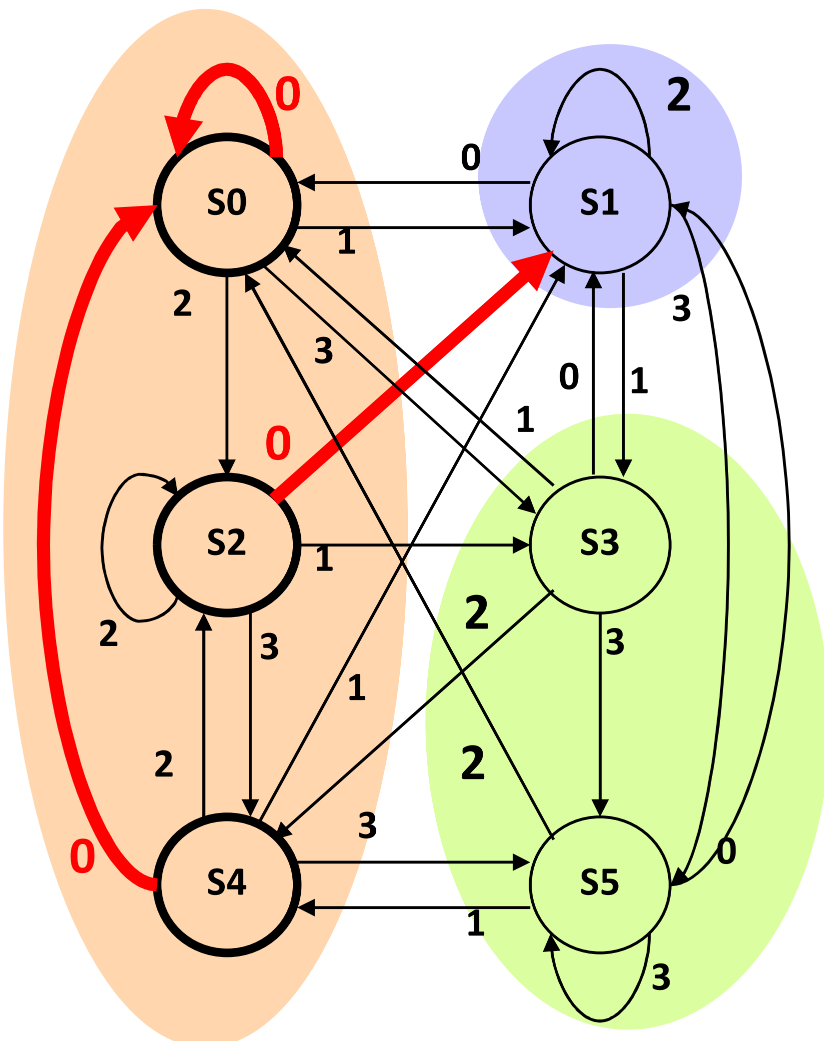

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state transition table

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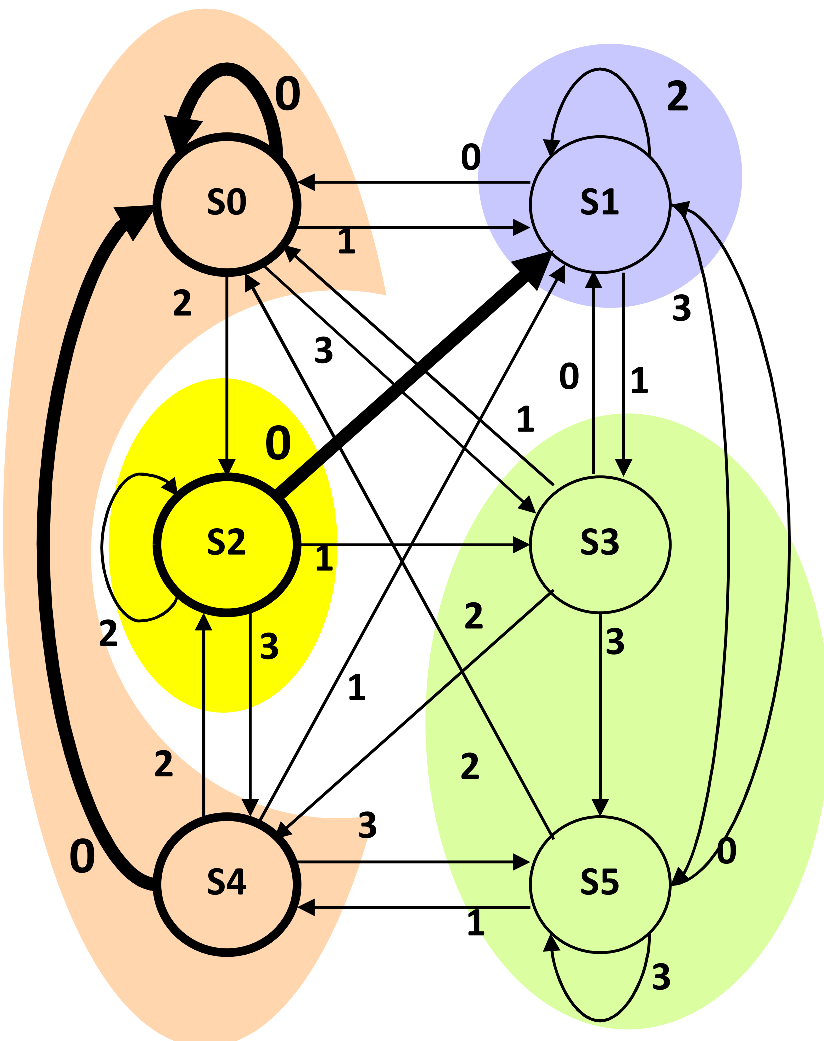

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state transition table

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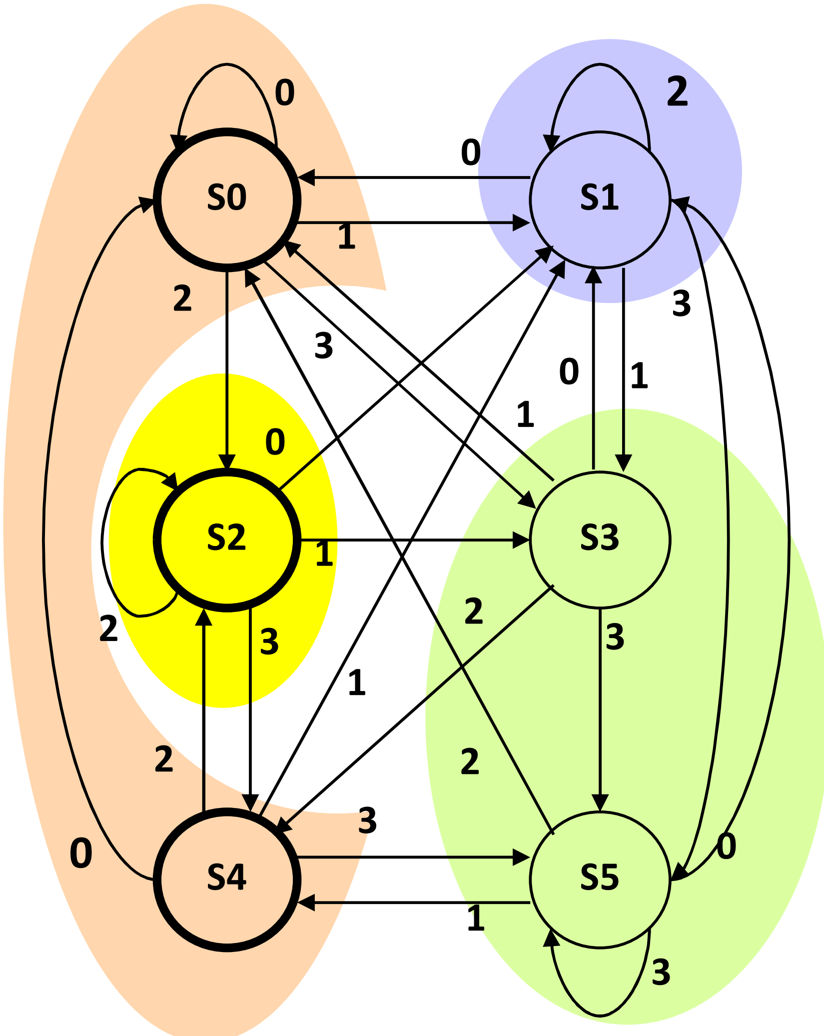

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S3	S1	S0	S4	S5	0
S4	S0	S1	S2	S5	1
S5	S1	S4	S0	S5	0

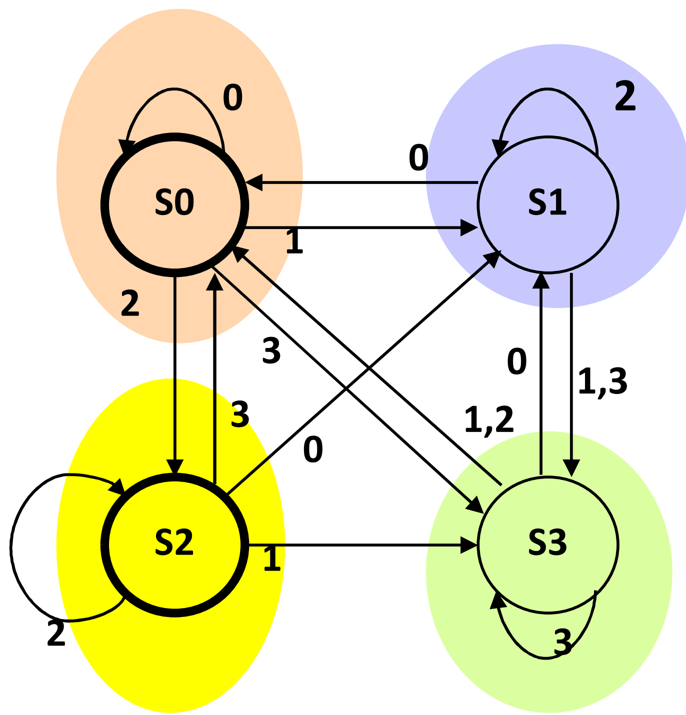
state transition table

Finally convert groups to states:

Can combine states S0-S4 and S3-S5.

In table replace all S4 with S0 and all S5 with S3

Minimized Machine



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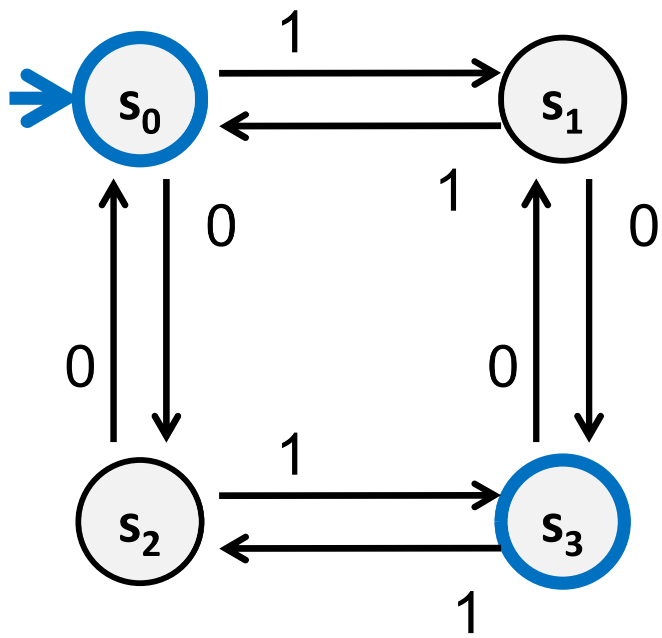
state
transition table

A Simpler Minimization Example



#0s is even

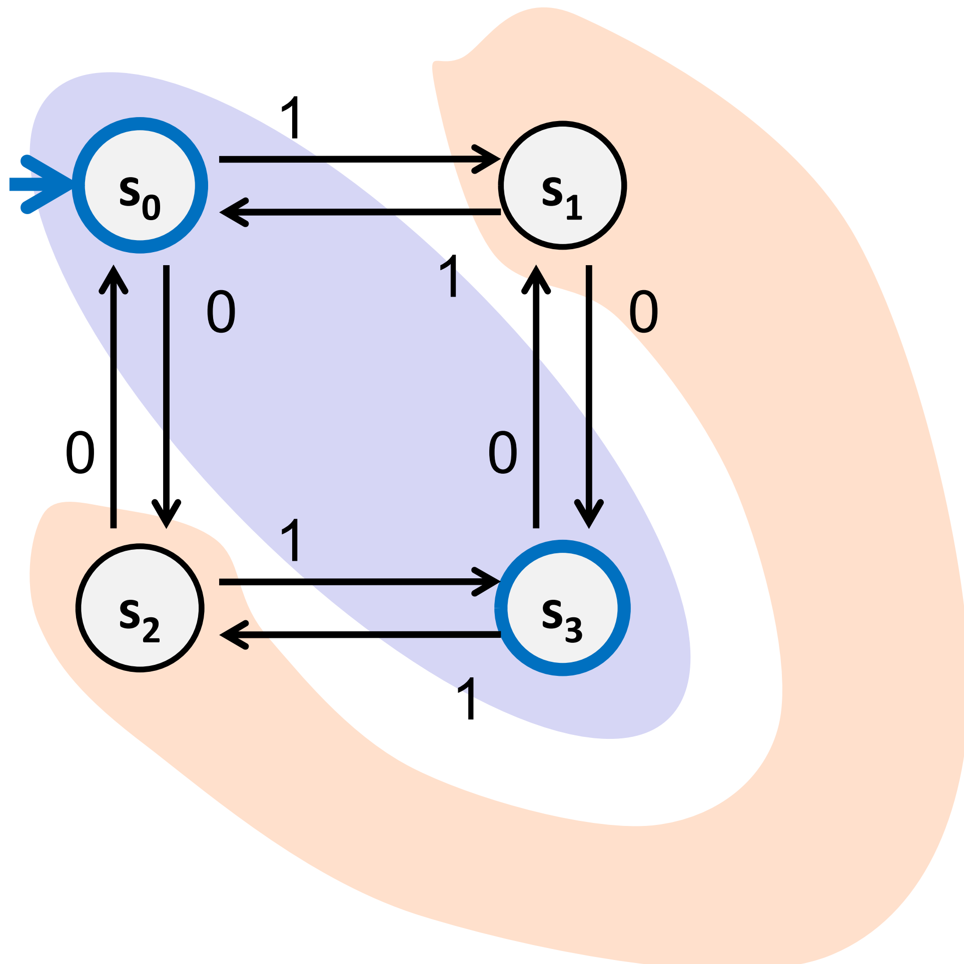
#0s is odd

#1s is even

#1s is odd

The set of all binary strings with # of 1's $\equiv$ # of 0's (mod 2).

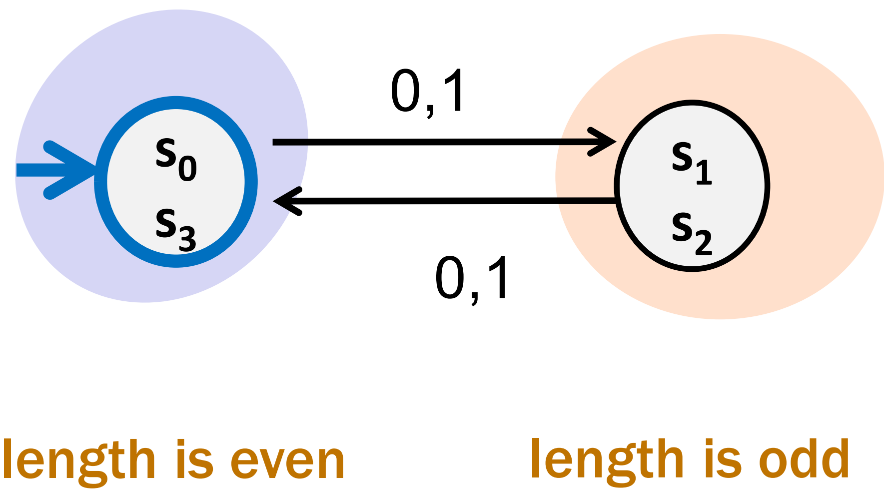
A Simpler Minimization Example



Split states into accept/reject groups

Every symbol causes the DFA to go from one group to the other so neither group needs to be split

Minimized DFA



The set of all binary strings with # of 1's ≡ # of 0's (mod 2).

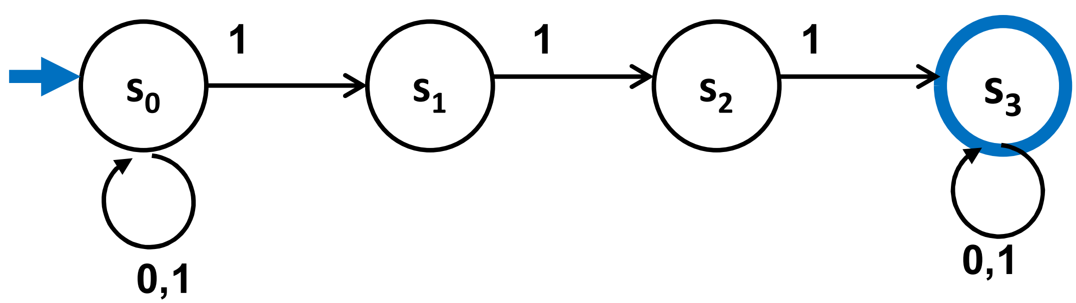
= The set of all binary strings with even length.

The Characters

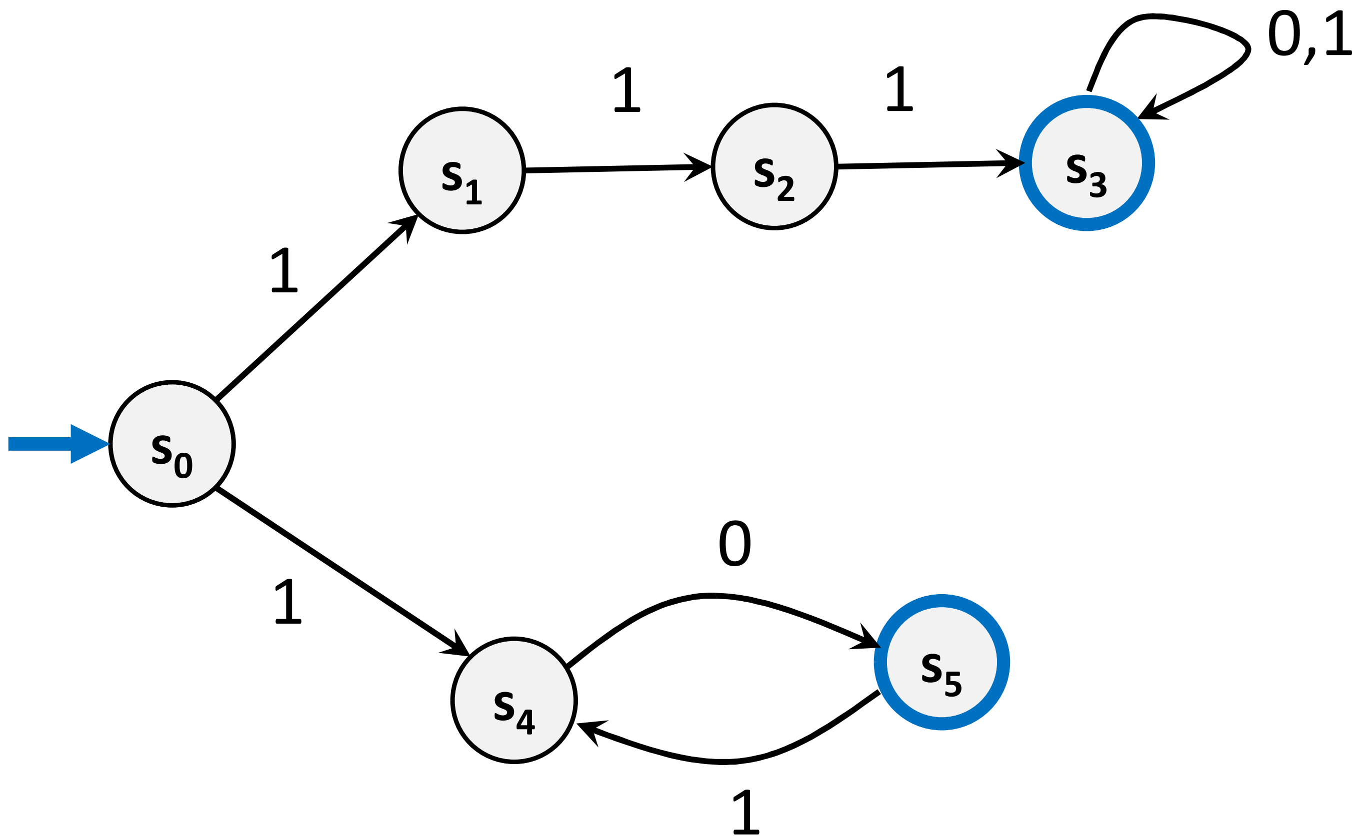
REs	$\subseteq$	CFGs
DFAs	$\subseteq$	NFAs

Nondeterministic Finite Automata (NFA)

- Graph with start state, final states, edges labeled by symbols (like DFA) but
  - Not required to have exactly 1 edge out of each state labeled by each symbol— can have 0 or >1
  - Also can have edges labeled by empty string ε
- Definition: x is in the language recognized by an NFA if and only if <u>some</u> valid execution of the machine gets to an accept state

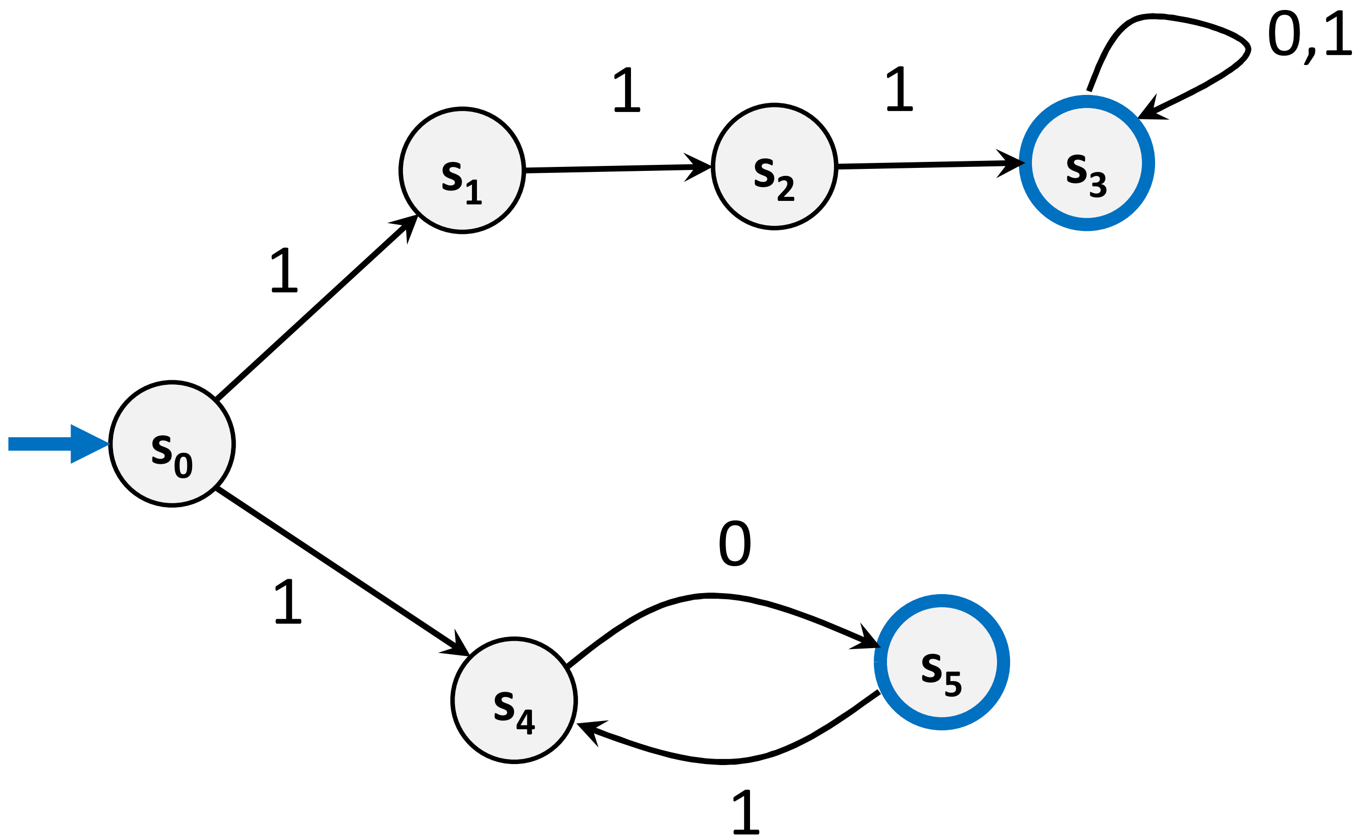


Consider This NFA



What language does this NFA accept?

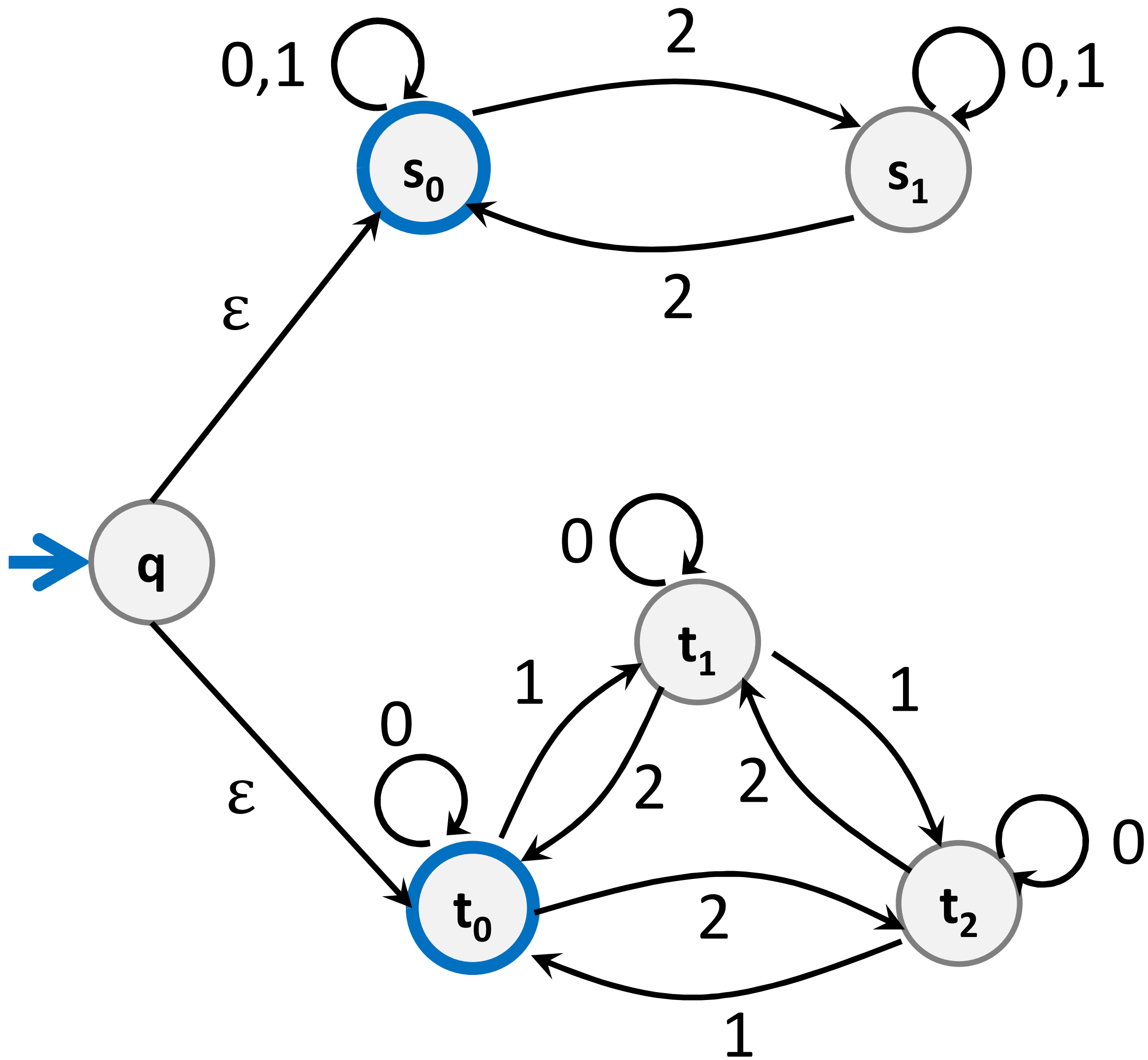
Consider This NFA



What language does this NFA accept?

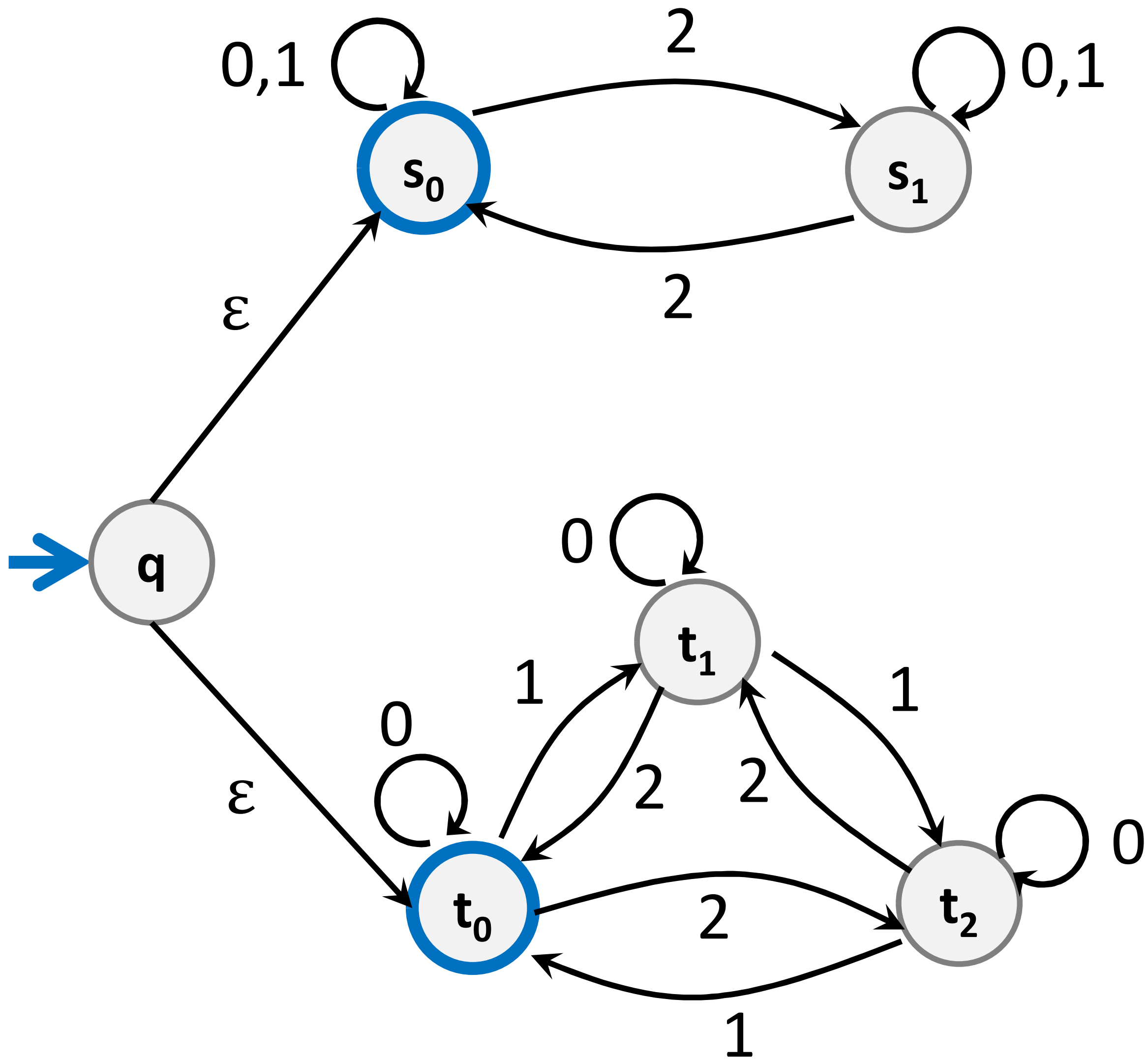
10(10)* U 111 (0 U 1)*

NFA ε-moves



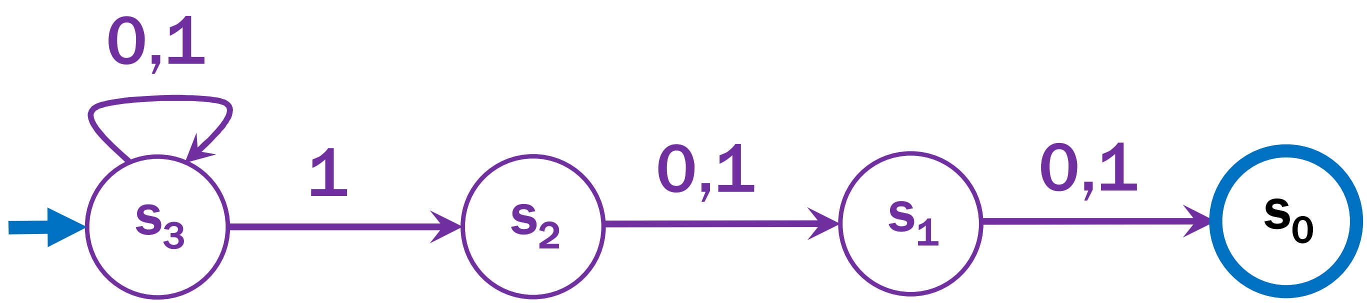
NFA ε-moves

Strings over {0,1,2} w/even # of 2's OR sum to 0 mod 3

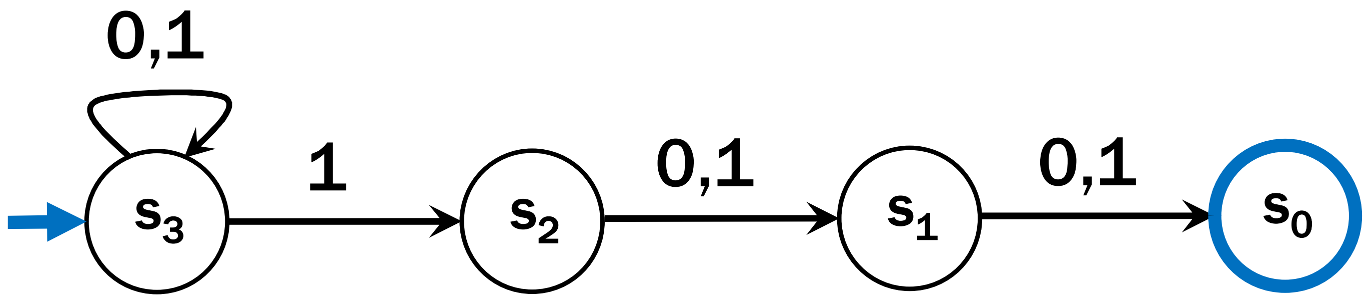


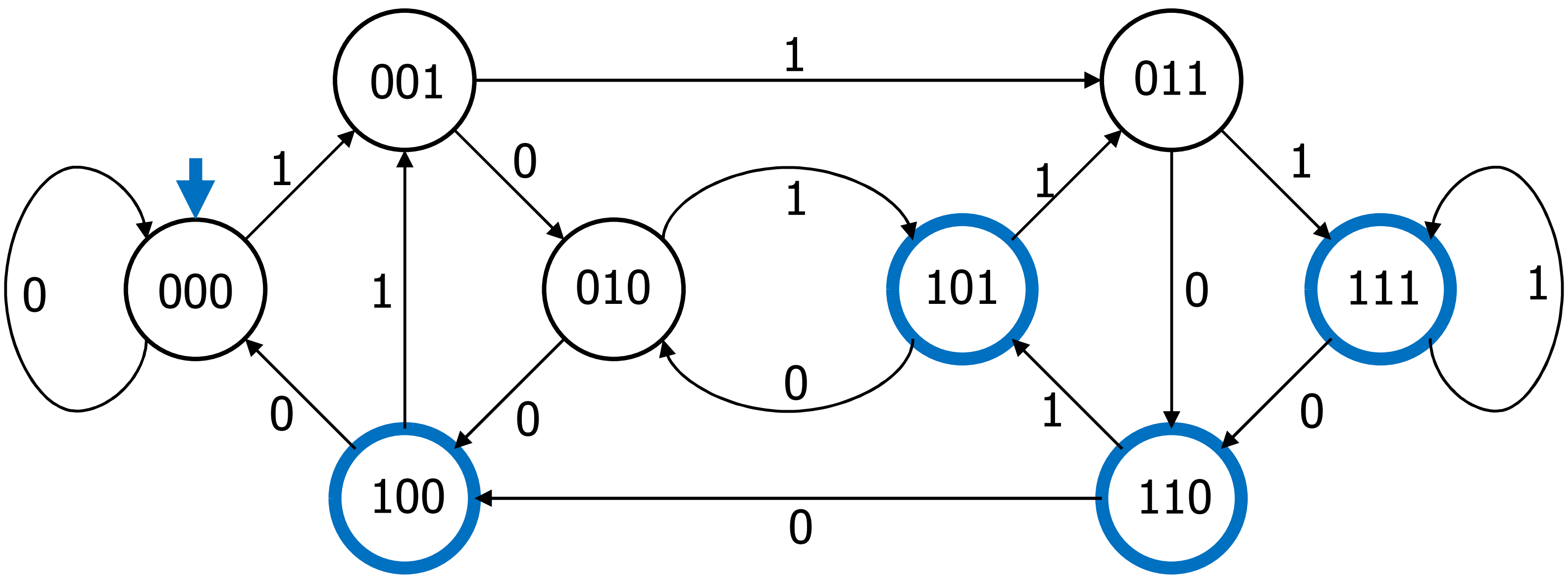
NFA for set of binary strings with a 1 in the 3rd position from the end

NFA for set of binary strings with a 1 in the 3rd position from the end



Compare with the smallest DFA





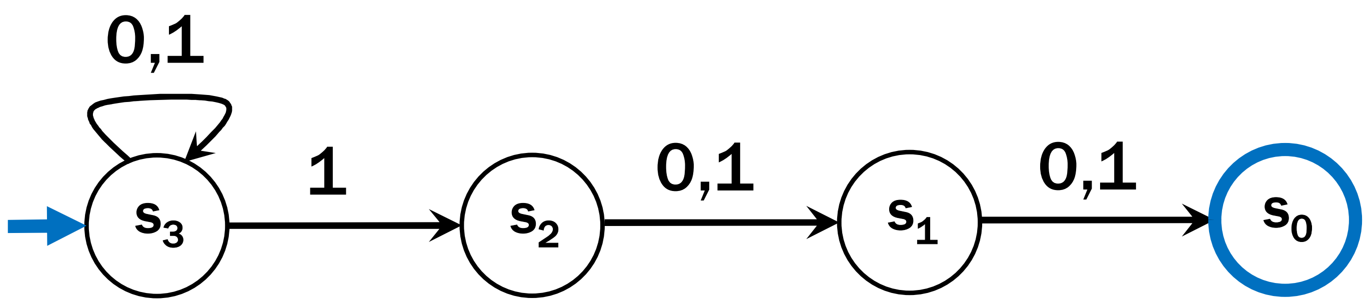
Three ways of thinking about NFAs

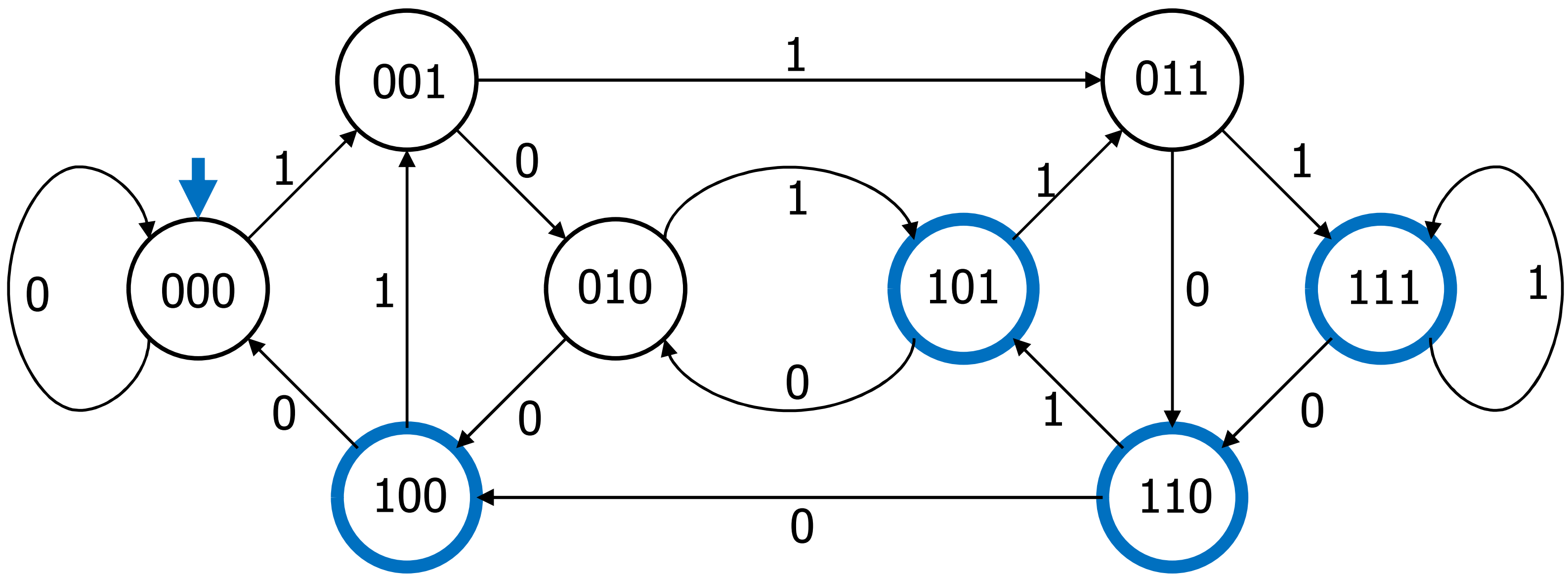
- Perfect guesser: The NFA has input x and whenever there is a choice of what to do it magically guesses a good one (if one exists)

- Parallel exploration: The NFA computation runs all possible computations on x step-by-step at the same time in parallel

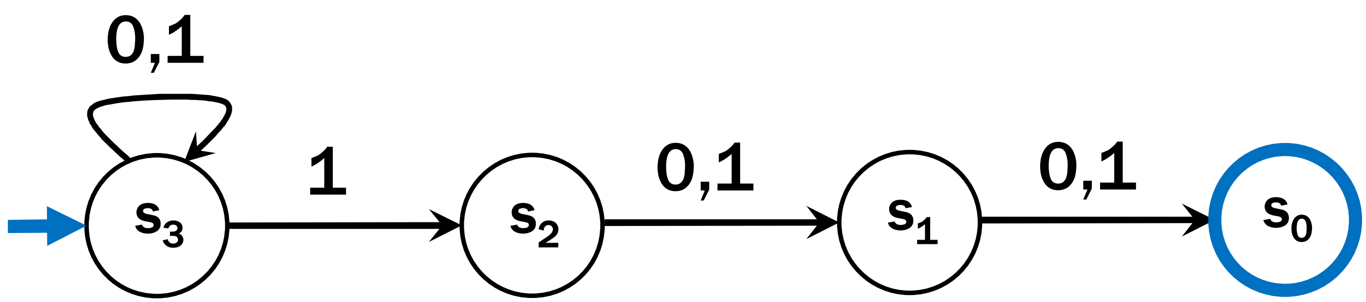
- Outside observer: Is there a path labeled by x from the start state to some accepting state?

Compare with the smallest DFA

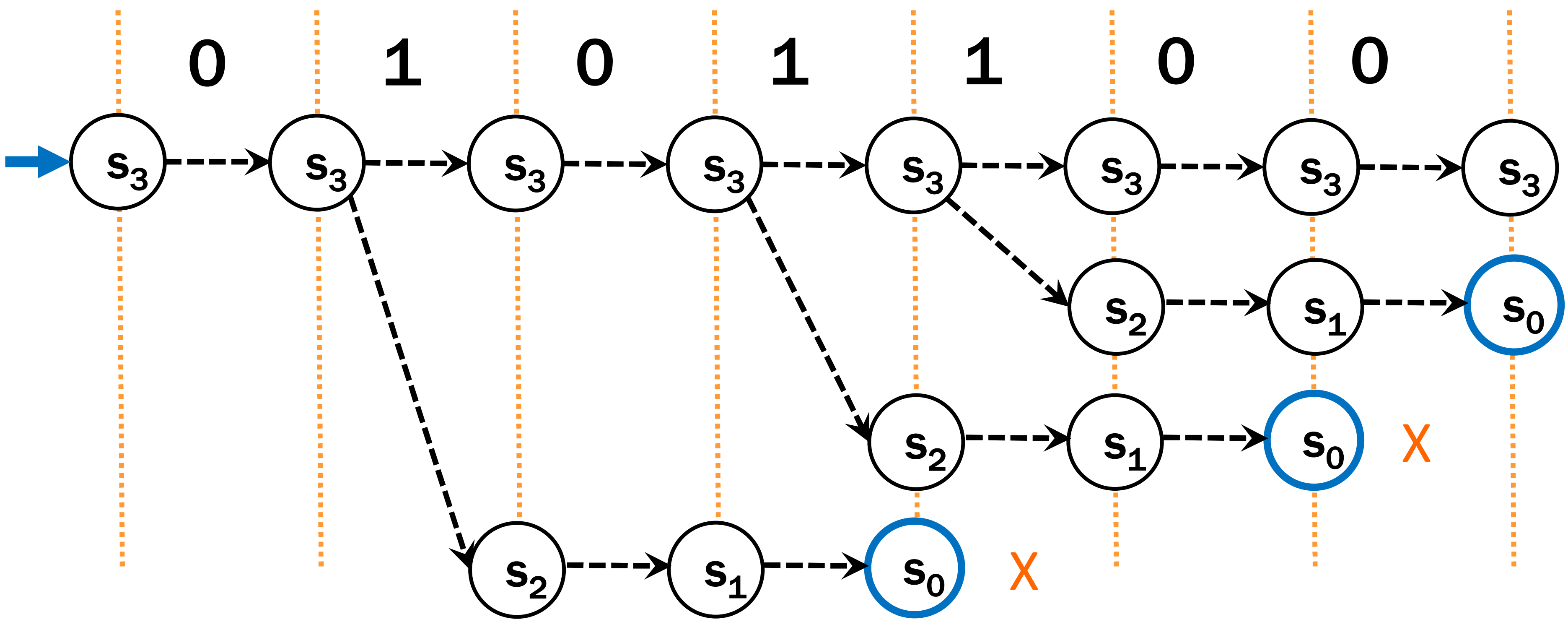




Parallel Exploration view of an NFA



Input string 0101100



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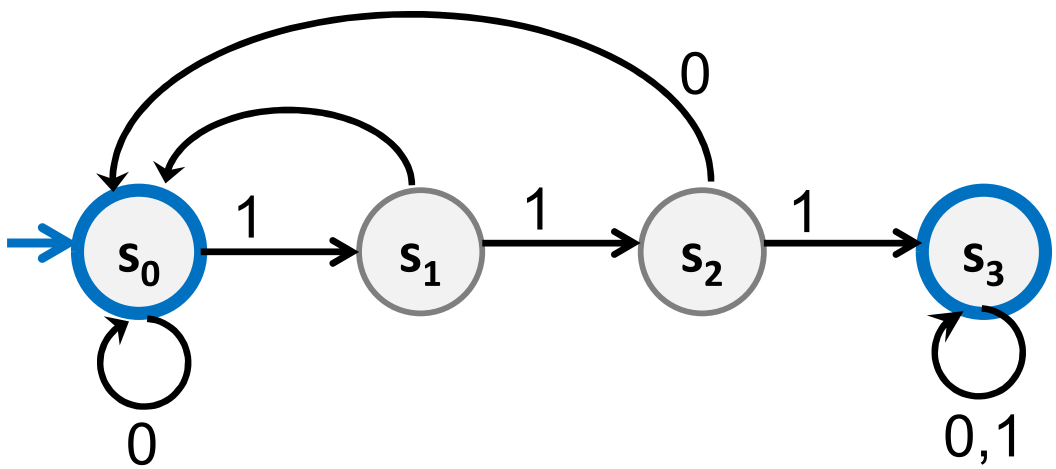
Path Labels

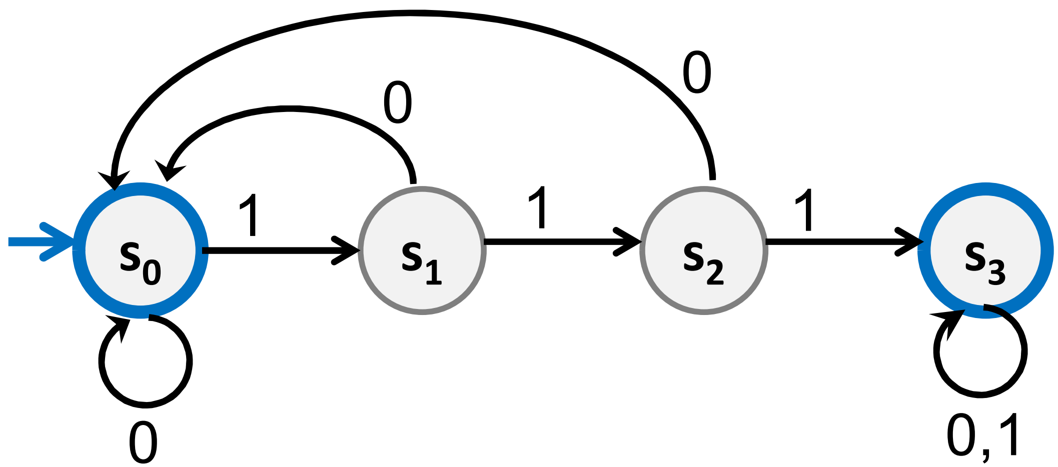
Def: The label of path $v_0, v_1, \ldots, v_n$ is the <u>concatenation</u> of the labels of the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$

Example: The label of path s_0, s_1, s_2, s_0, s_0 is 1100



Deterministic Finite Automata (DFA)

- **Theorem:** x is in the language recognized by an DFA if and only if x labels a path from the start state to some final state

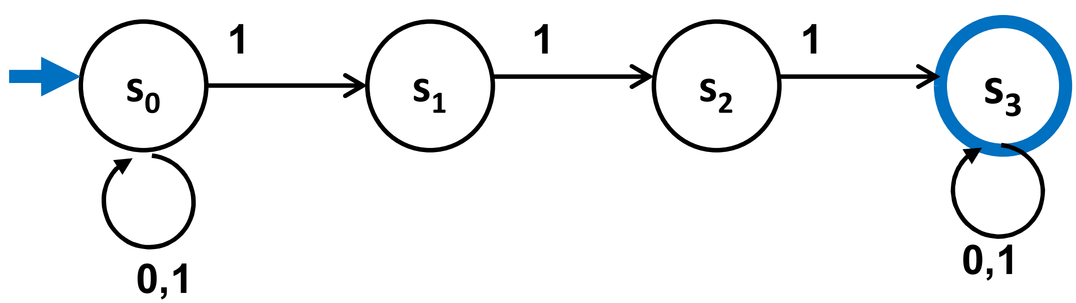


- Path $v_0, v_1, \ldots, v_n$ with $v_0 = s_0$ and label x describes a correct simulation of the DFA on input x
  - i-th step must match the i-th character of x

Nondeterministic Finite Automata (NFA)

- Graph with start state, final states, edges labeled by symbols (like DFA) but
  - Not required to have exactly 1 edge out of each state labeled by each symbol— can have 0 or >1
  - Can also have edges labeled by empty string ε

- **Theorem:** x is in the language recognized by an NFA if and only if x labels <u>some</u> path from the start state to an accepting state



Summary of NFAs

- Generalization of DFAs
  - drop two restrictions of DFAs
  - every DFA <u>is</u> an NFA

- *Seem* to be more powerful
  - designing is easier than with DFAs

- *Seem* related to regular expressions

The story so far...

$$\boxed{\text{REs}} \subseteq \boxed{\text{CFGs}}$$

$$\boxed{\text{DFAs}} \subseteq \boxed{\text{NFAs}}$$

NFAs and regular expressions

Theorem: For any set of strings (language) A described by a regular expression, there is an NFA that recognizes A.

Proof idea: Structural induction based on the recursive definition of regular expressions...

Regular Expressions over Σ

- **Basis:**
  – ε is a regular expression
  – *a* is a regular expression for any $a \in \Sigma$

- **Recursive step:**
  – If A and B are regular expressions, then so are:

    A $\cup$ B

    AB

    A*

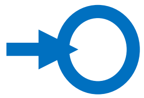
Base Case

- **Case ε:**

- **Case *a*:**

Base Case

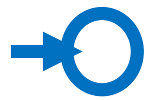
- **Case ε:**



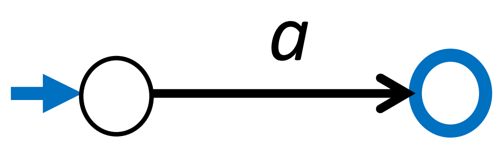
- **Case *a*:**

Base Case

- **Case ε:**



- **Case *a*:**



Regular Expressions over Σ

- **Basis:**
  - ε is a regular expression
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- **Recursive step:**
  - If A and B are regular expressions, then so are:

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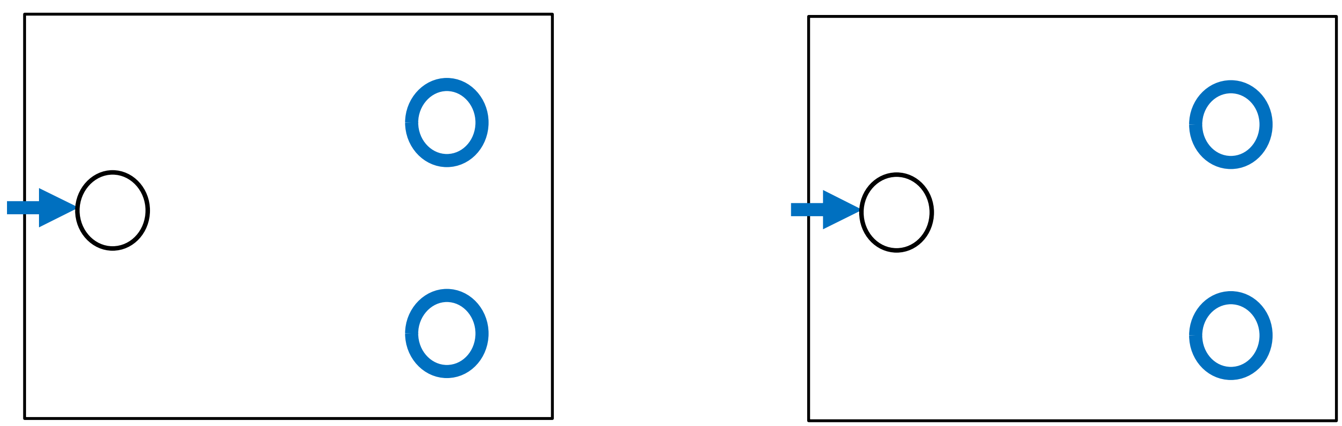
Inductive Hypothesis

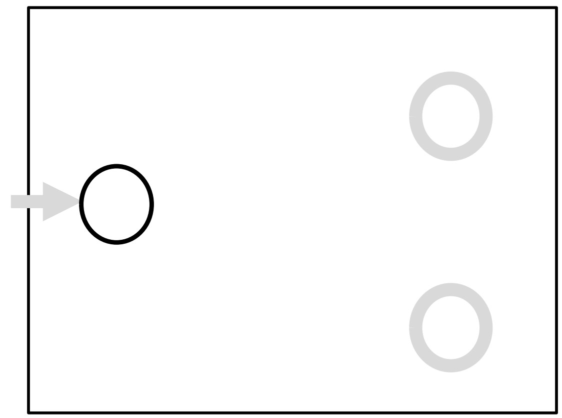
- Suppose that for some regular expressions A and B there exist NFAs N_A and N_B such that N_A recognizes the language given by A and N_B recognizes the language given by B



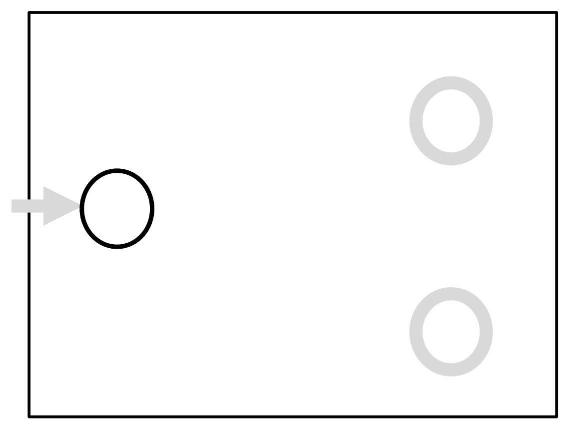
N_A N_B

Inductive Step

Case A ∪ B:



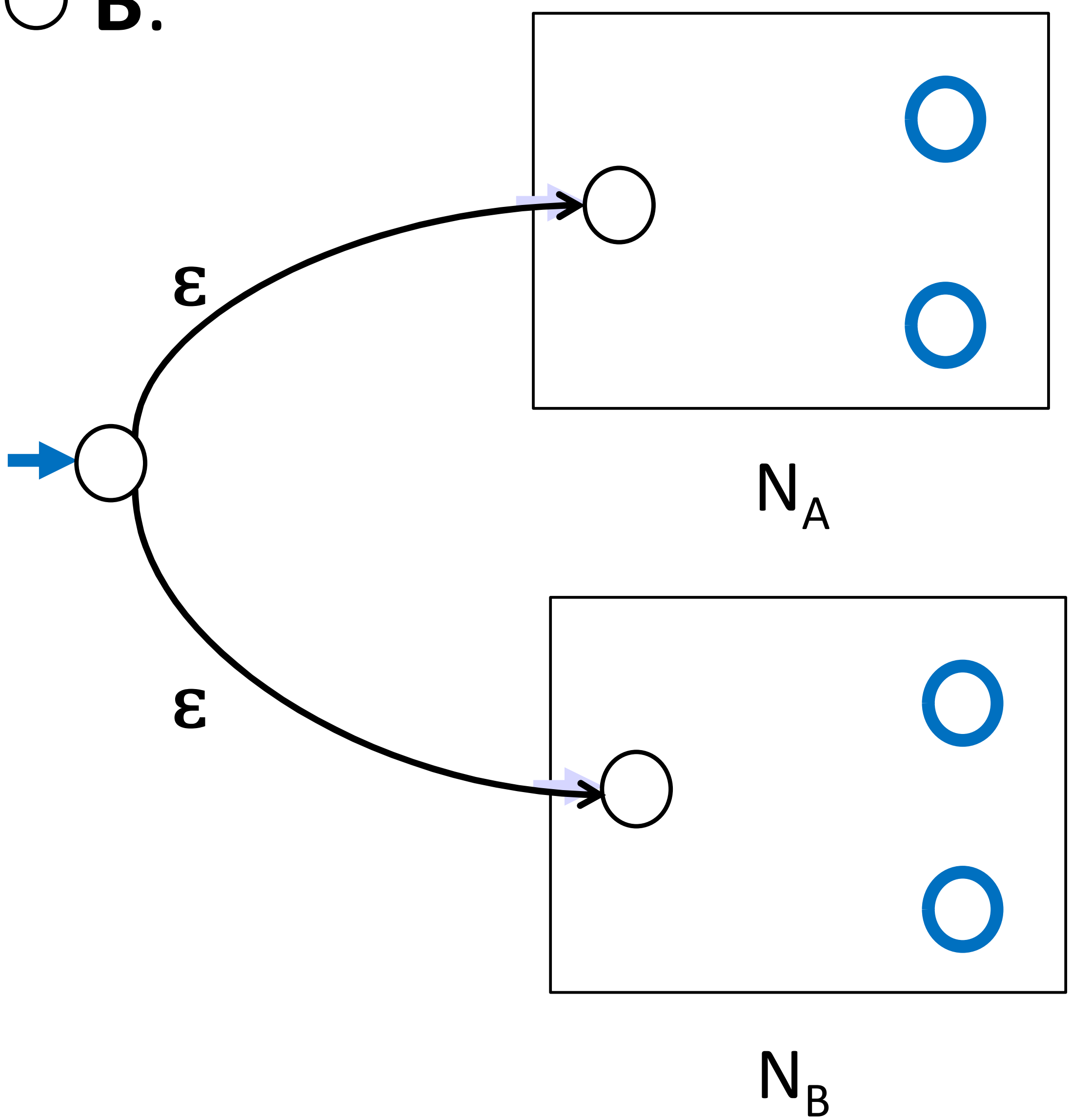
N_A



N_B

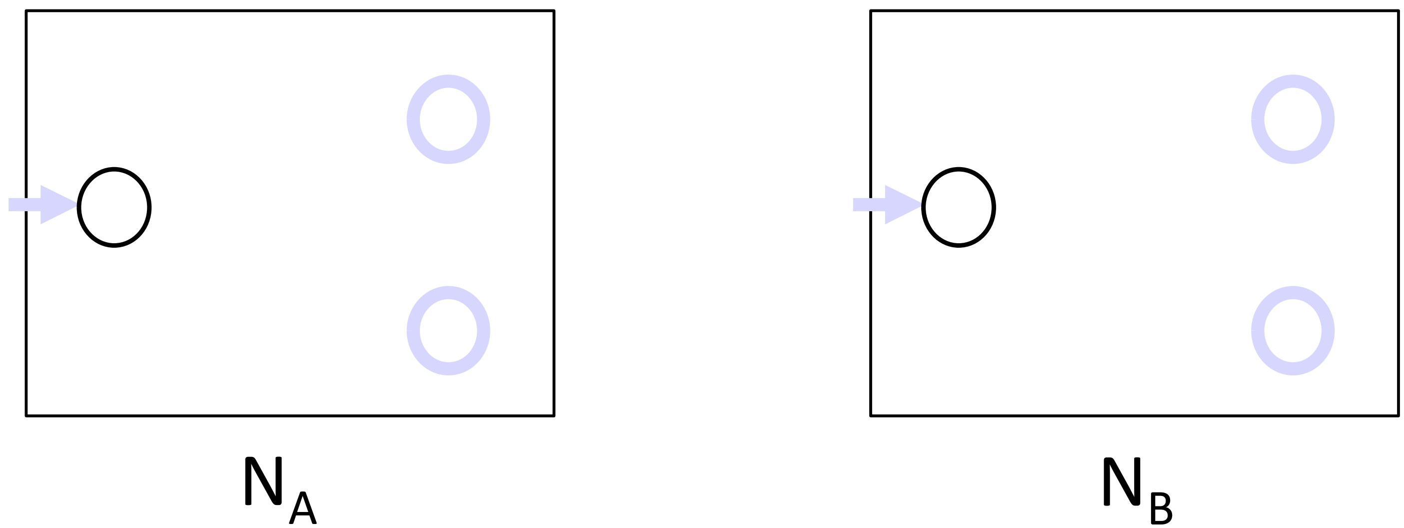
Inductive Step

Case A ∪ B:



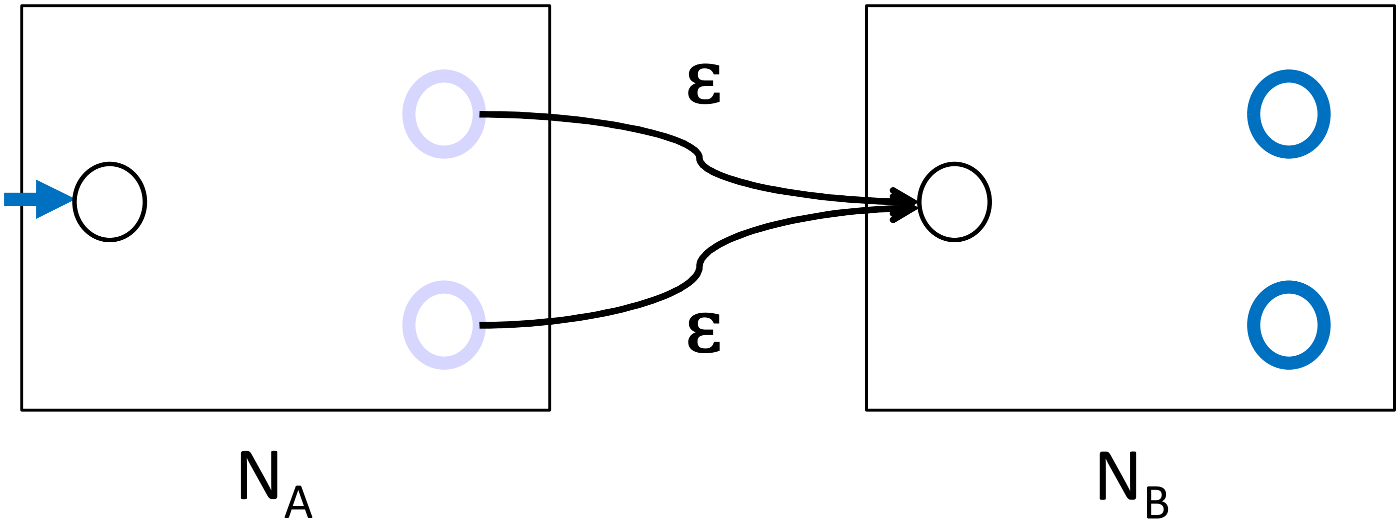
Inductive Step

Case AB:



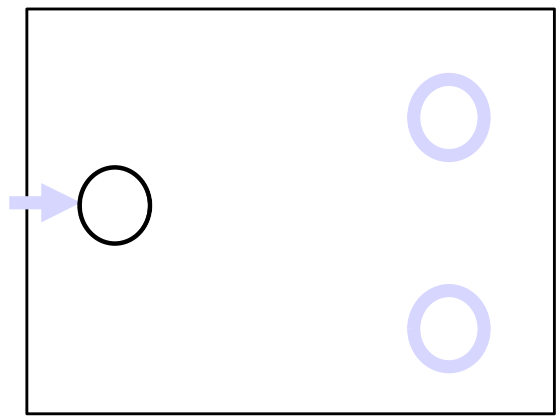
Inductive Step

Case AB:



Inductive Step

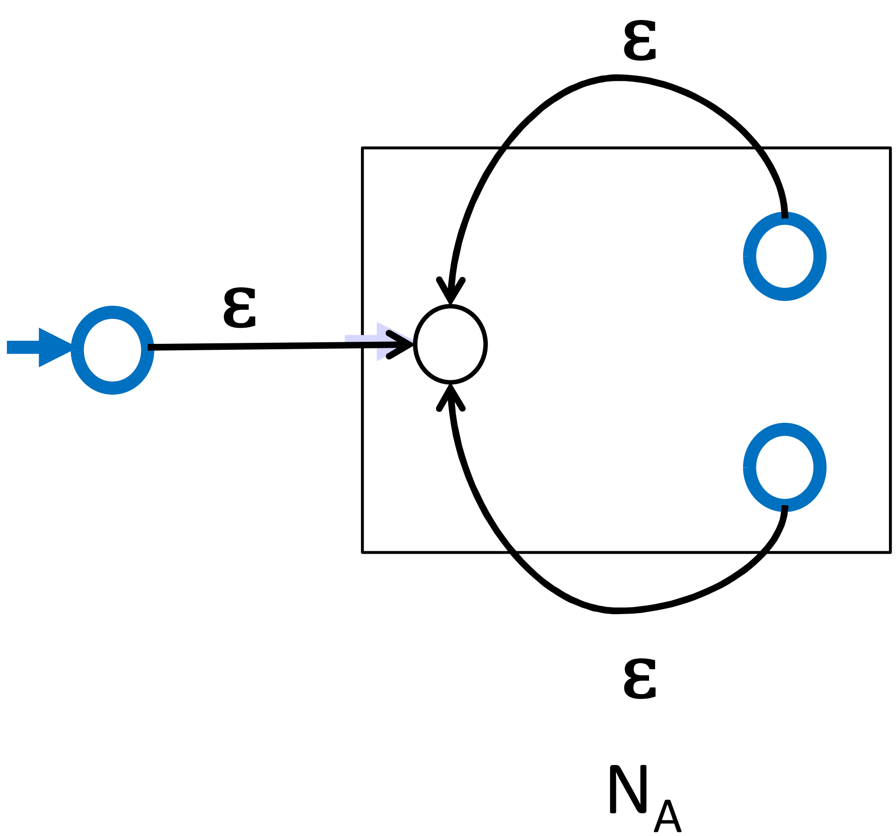
Case A*



N_A

Inductive Step

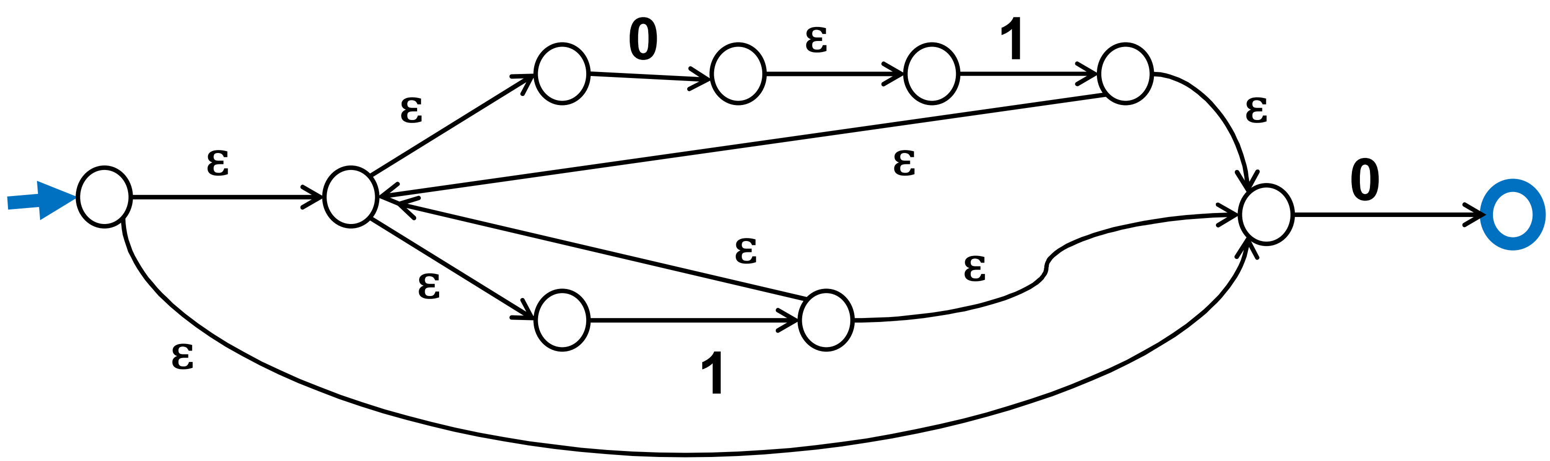
Case A*



Build an NFA for $(01 \cup 1)^*0$

Solution

(01 ∪1)*0



The story so far...

$$
\begin{array}{ccc}
\text{REs} & \subseteq & \text{CFGs} \\
 & \rotatebox{-45}{$\subseteq$} & \\
\text{DFAs} & \subseteq & \text{NFAs}
\end{array}
$$

NFAs and DFAs

Every DFA is an NFA

- DFAs have requirements that NFAs don't have

Can NFAs recognize more languages?

NFAs and DFAs

Every DFA is an NFA

– DFAs have requirements that NFAs don't have

Can NFAs recognize more languages? No!

Theorem: For every NFA there is a DFA that recognizes exactly the same language

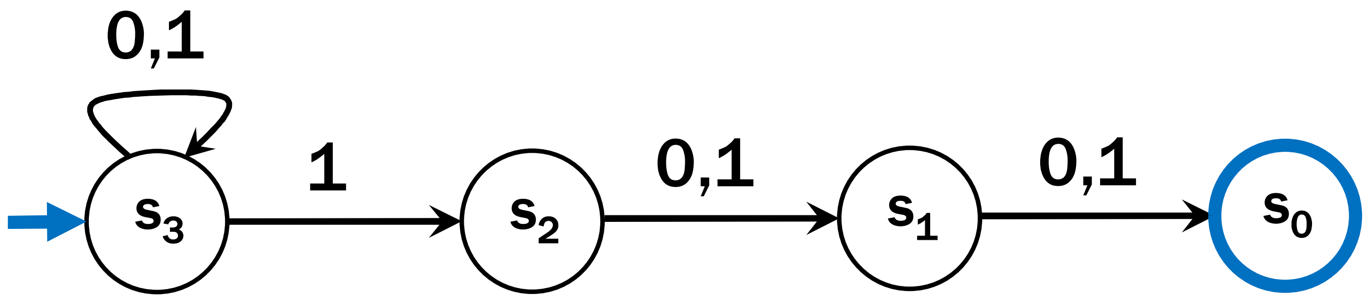
Three ways of thinking about NFAs

- Perfect guesser: The NFA has input x and whenever there is a choice of what to do it magically guesses a good one (if one exists)

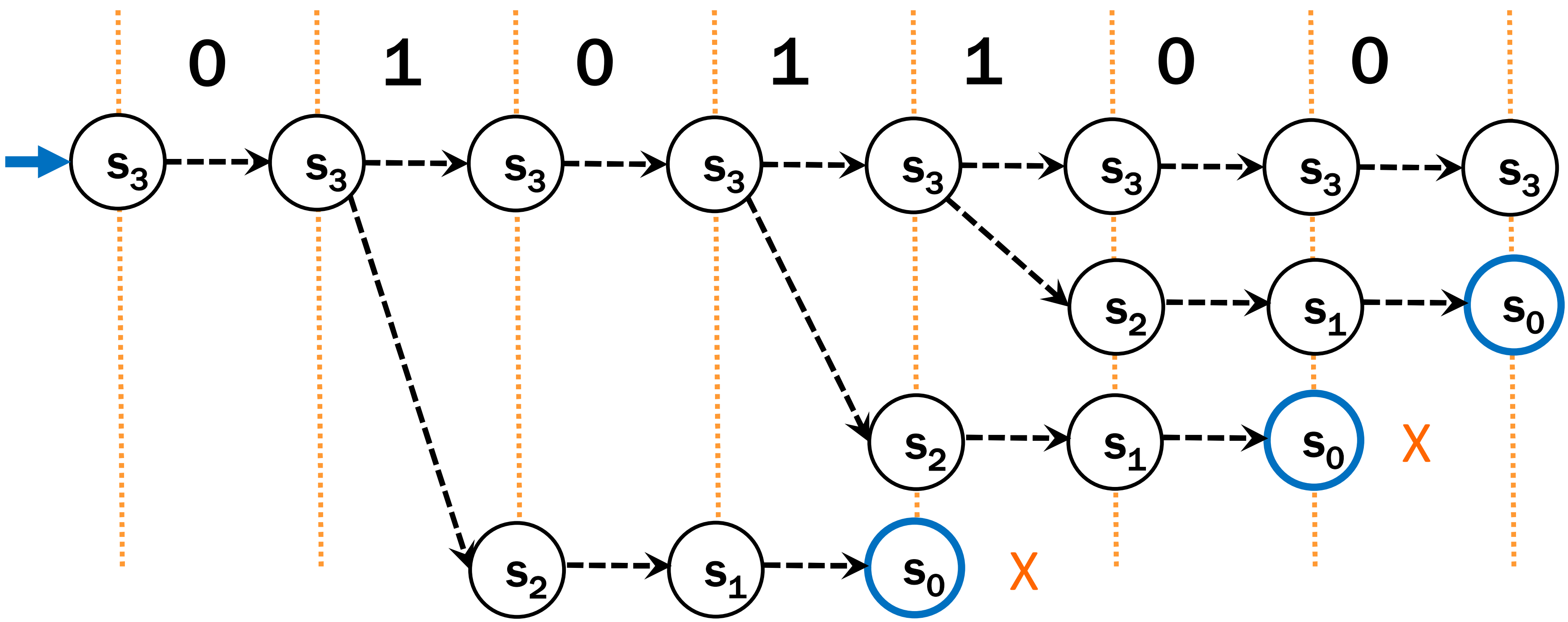
- Parallel exploration: The NFA computation runs all possible computations on x step-by-step at the same time in parallel

- Outside observer: Is there a path labeled by x from the start state to some final state?

Parallel Exploration view of an NFA



Input string 0101100



Conversion of NFAs to a DFAs

- **Construction Idea:**
  - The DFA keeps track of ALL states reachable in the NFA along a path labeled by the input so far

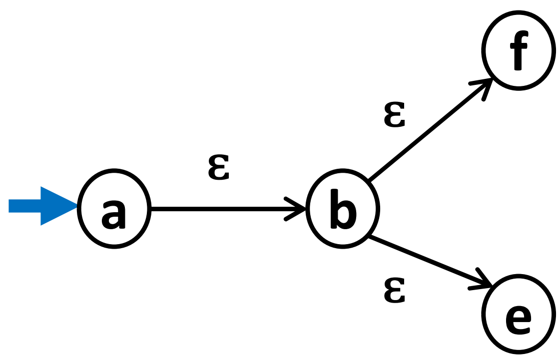
    (Note: not all *paths*; all *last states* on those paths.)

  - There will be one state in the DFA for each *subset* of states of the NFA that can be reached by some string

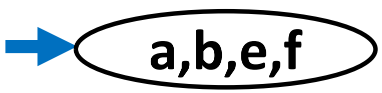
Conversion of NFAs to a DFAs

New start state for DFA

– The set of all states reachable from the start state of the NFA using only edges labeled ε



NFA

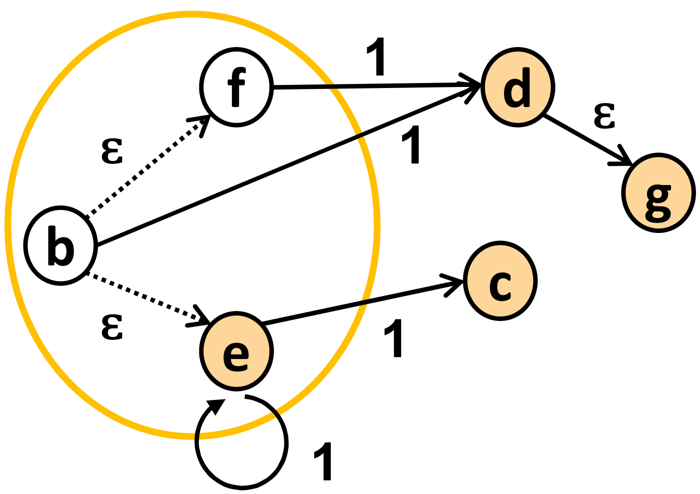


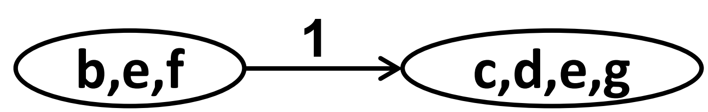
DFA

Conversion of NFAs to a DFAs

For each state of the DFA corresponding to a set S of states of the NFA and each symbol s

- **Add an edge labeled s to state corresponding to T, the set of states of the NFA reached by**
  - starting from some state in S, then
  - following one edge labeled by s, and
    then following some number of edges labeled by **ε**
- T **will be ∅ if no edges from** S **labeled** s **exist**





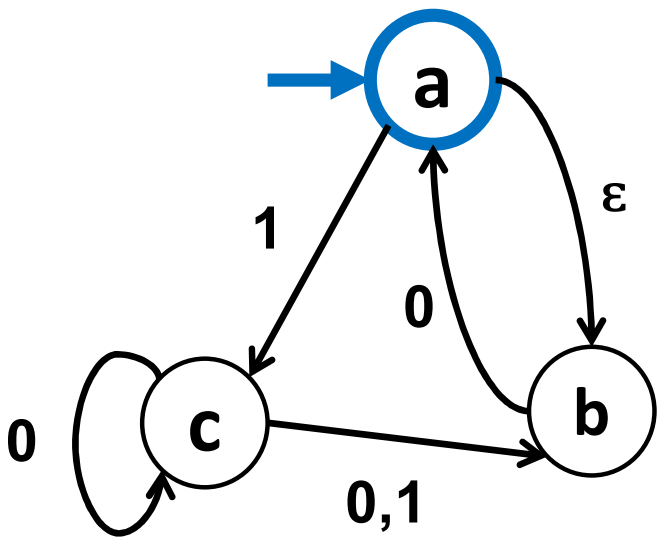
Conversion of NFAs to a DFAs

Final states for the DFA

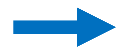
– All states whose set contain some final state of the NFA

NFA DFA

Example: NFA to DFA

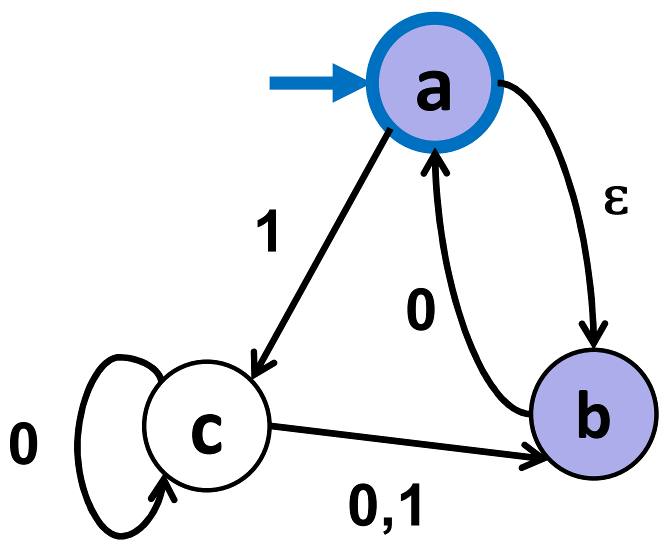


NFA

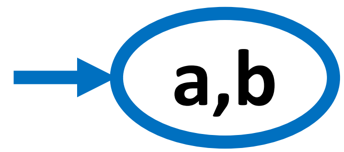


DFA

Example: NFA to DFA

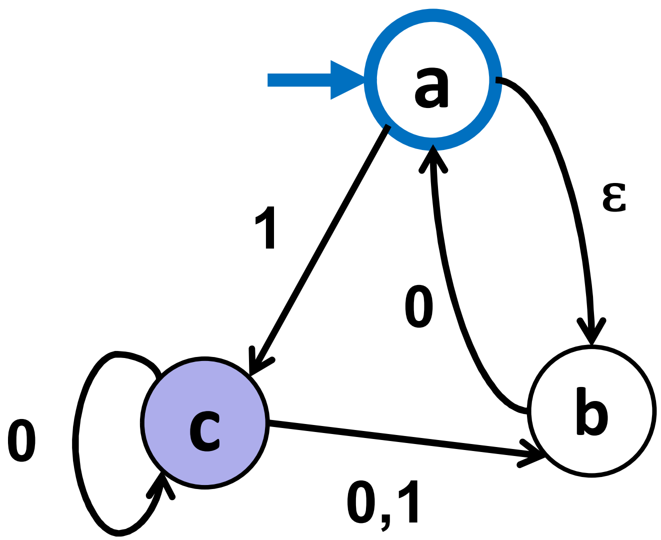


NFA

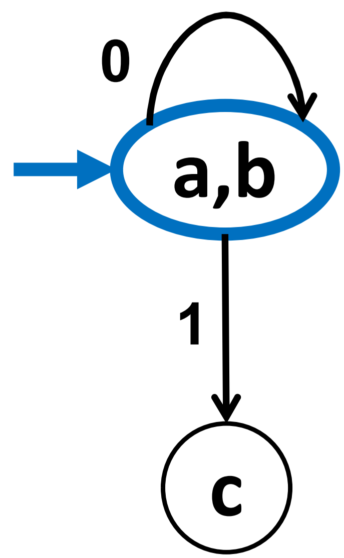


DFA

Example: NFA to DFA

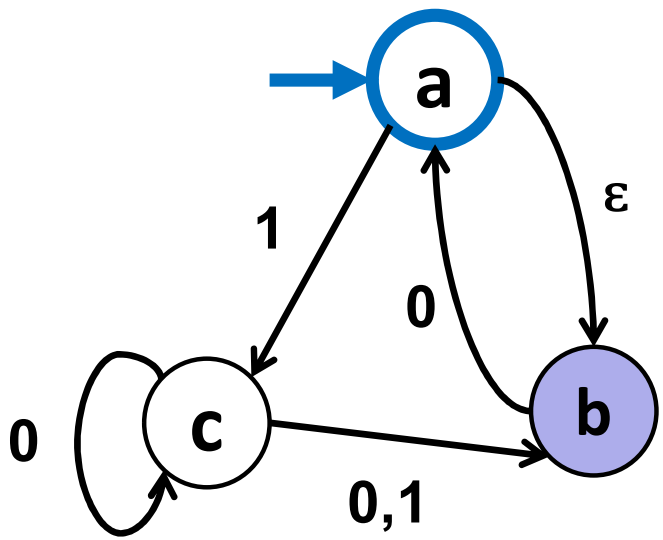


NFA

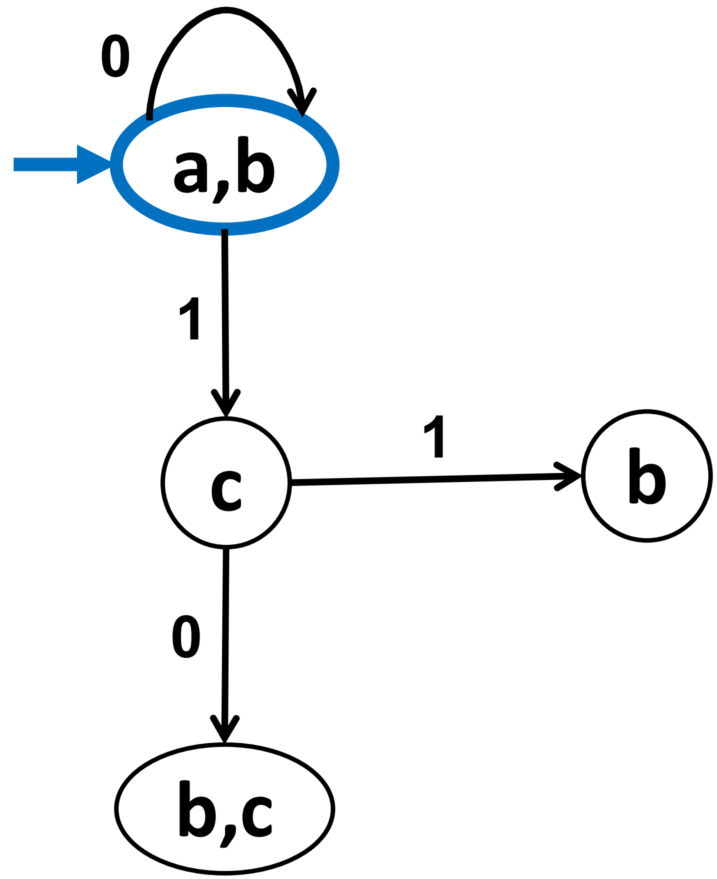


DFA

Example: NFA to DFA

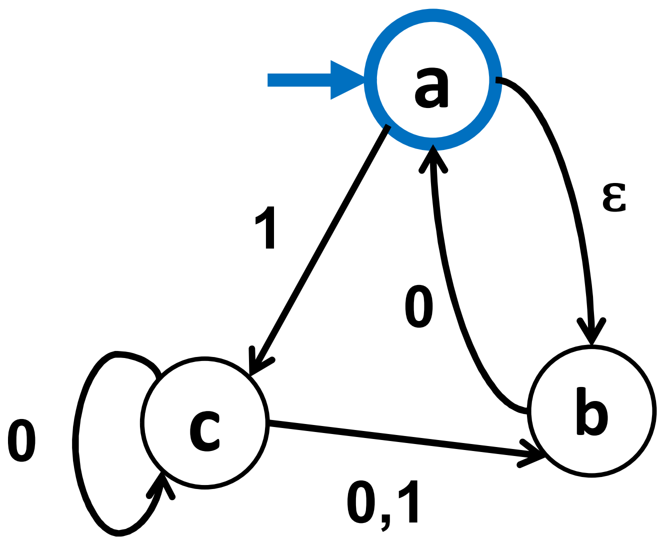


NFA

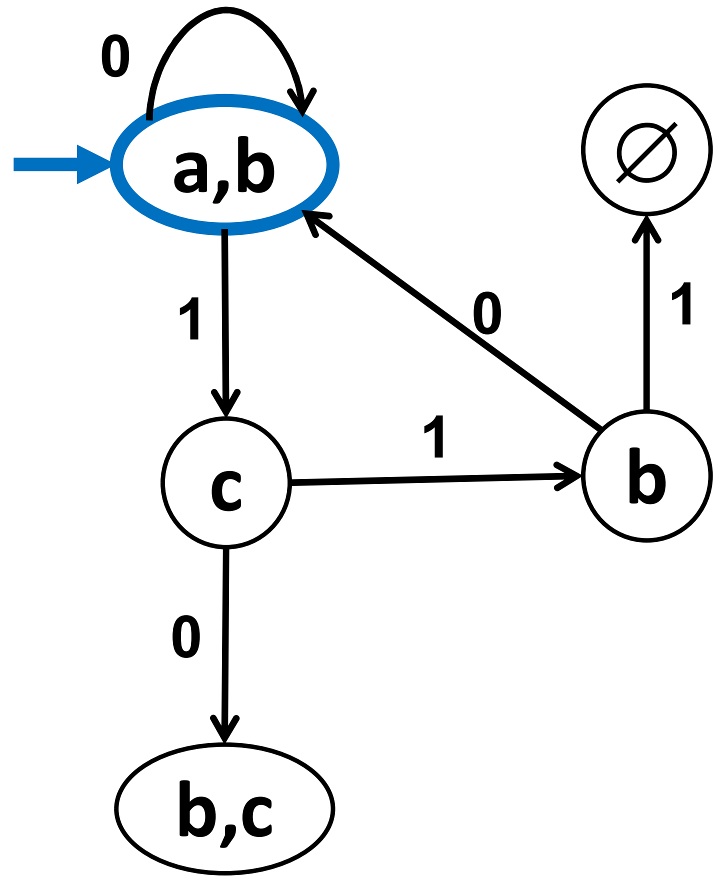


DFA

Example: NFA to DFA

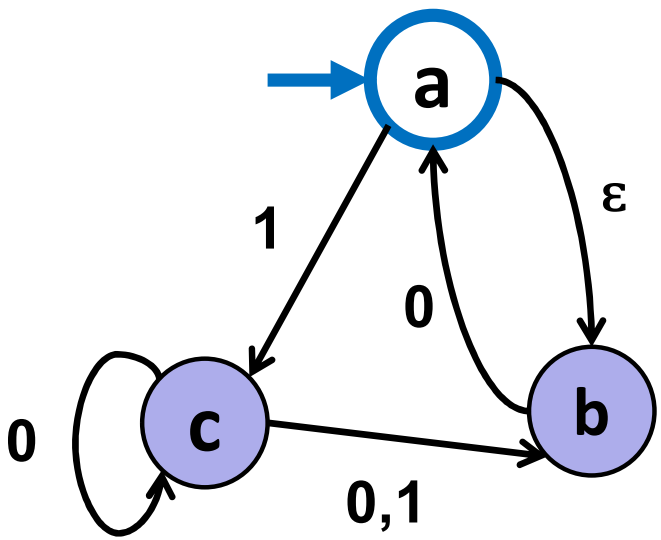


NFA

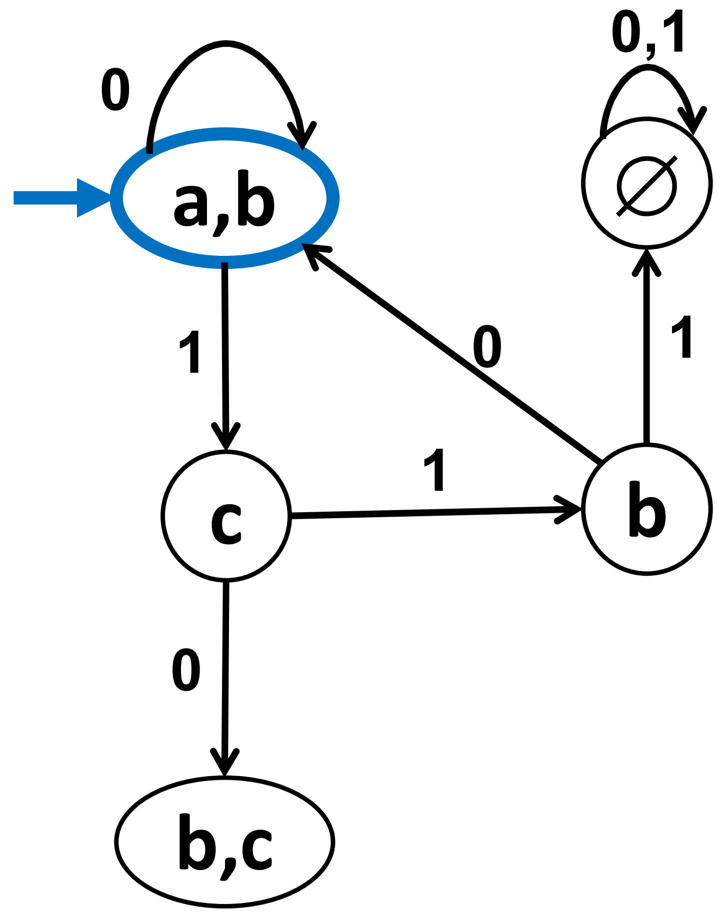


DFA

Example: NFA to DFA

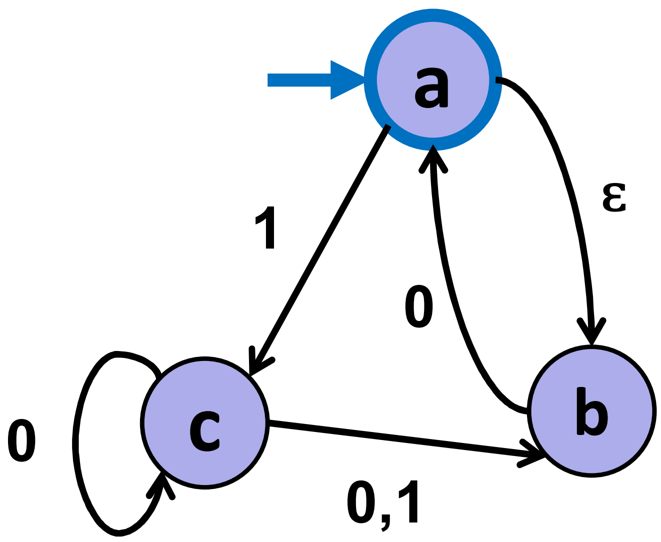


NFA

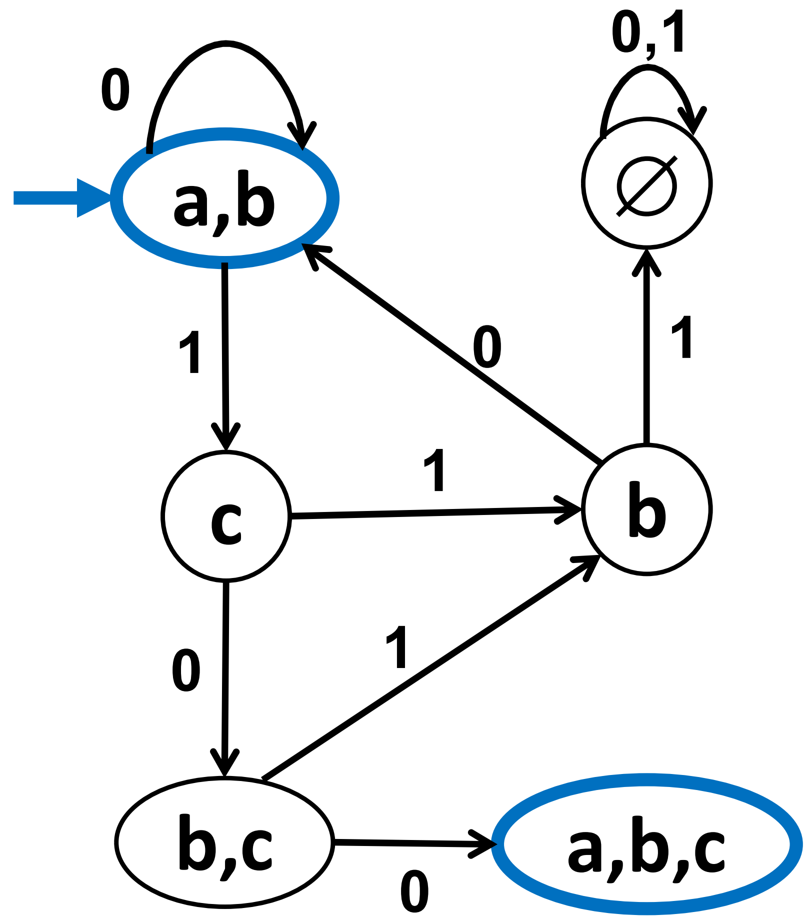


DFA

Example: NFA to DFA

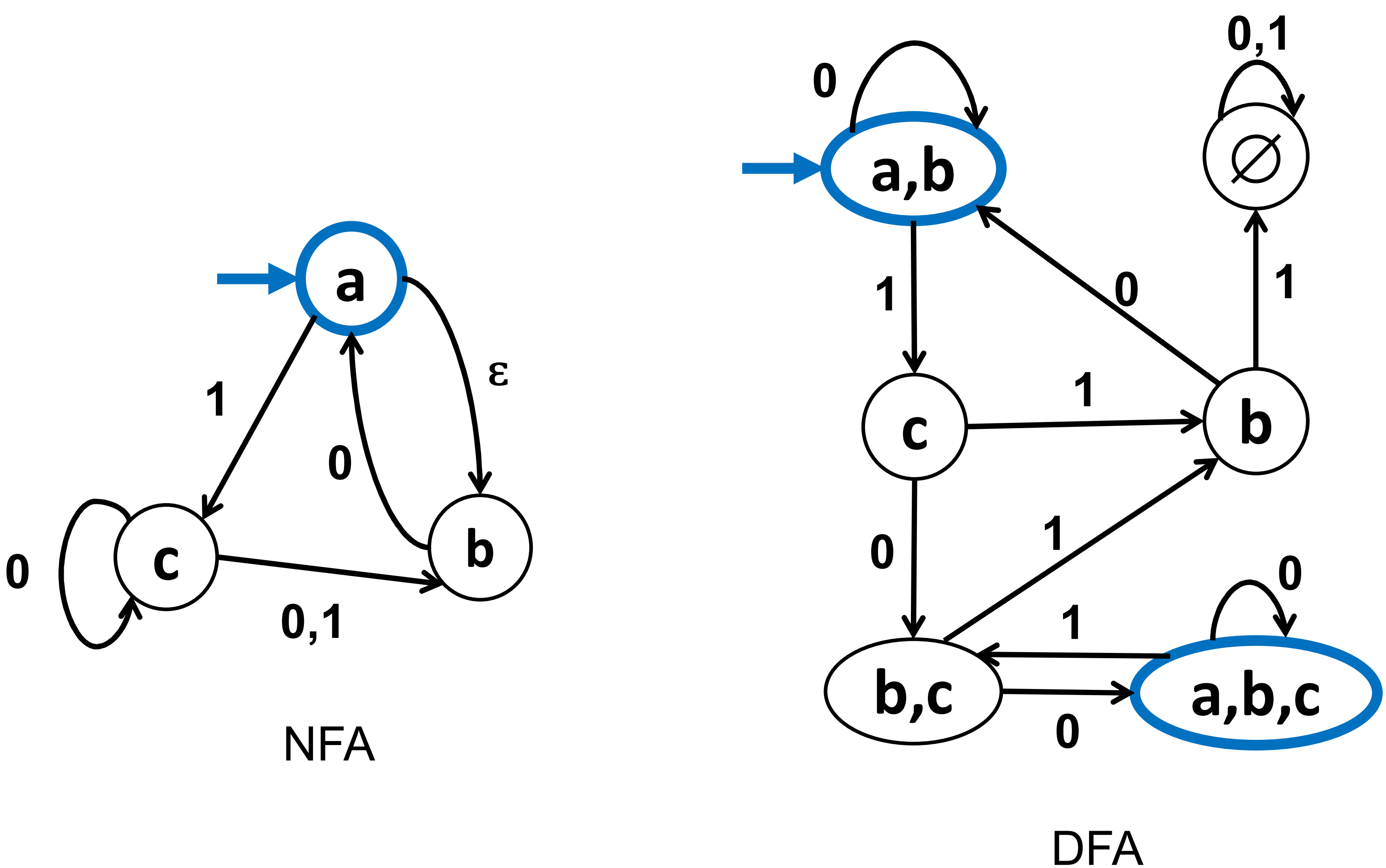


NFA



DFA

Example: NFA to DFA



The story so far...

REs	$\subseteq$	**CFGs**
	$\subseteq$	
DFAs	$=$	**NFAs**

Regular expressions $\subseteq$ NFAs $\equiv$ DFAs

We have shown how to build an optimal DFA for every regular expression

- Build NFA
- Convert NFA to DFA using subset construction
- Minimize resulting DFA

Thus, we could now implement a RegExp library

- most RegExp libraries actually simulate the NFA
- (even better: one can combine the two approaches: apply DFA minimization lazily while simulating the NFA)

The story so far...

REs $\subseteq$ **CFGs**

$\subseteq$

DFAs = **NFAs**

Is this $\subseteq$ really "=" or "$\subsetneq$"?

Regular expressions ≡ NFAs ≡ DFAs

Theorem: For any NFA, there is a regular expression that accepts the same language

Corollary: A language is recognized by a DFA (or NFA) if and only if it has a regular expression

You need to know these facts

– the construction for the Theorem is included in the slides after this, but you will not be tested on it

The story so far…

$$\begin{array}{ccc} \boxed{\text{REs}} & \subseteq & \boxed{\text{CFGs}} \\ \equiv & & \\ \boxed{\text{DFAs}} & \equiv & \boxed{\text{NFAs}} \end{array}$$

Languages represented by DFA, NFAs, or regular expressions are called Regular Languages

Recall: Algorithms for Regular Languages

We have seen algorithms for

- RE to NFA
- NFA to DFA
- DFA/NFA to RE (not tested)
- DFA minimization

Practice three of these in HW.

(May also be on the final.)

Example Corollary of These Results

Corollary: If $\mathbf{A}$ is the language of a regular expression, then $\overline{\mathbf{A}}$ is the language of a regular expression*.

(This is the complement with respect to the universe of all strings over the alphabet, i.e., $\overline{\mathbf{A}} = \mathbf{\Sigma}^* \setminus \boldsymbol{A}$.)

The story so far…

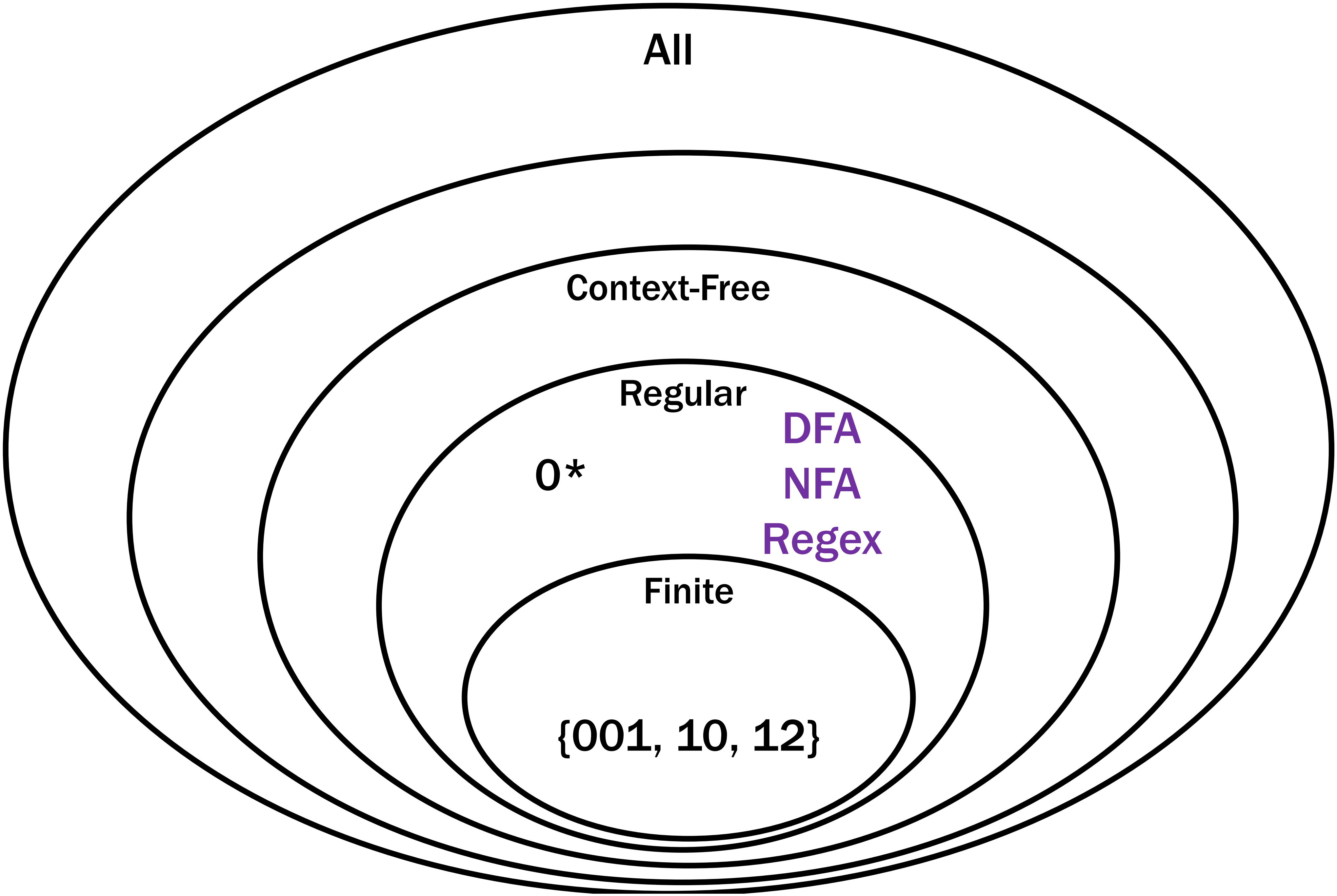
$$\begin{array}{ccc} \textbf{REs} & \subseteq & \textbf{CFGs} \\ \| & & \\ \textbf{DFAs} & = & \textbf{NFAs} \end{array}$$

<u>Now</u>: Is this $\subseteq$ really "$=$" or "$\subsetneq$"?

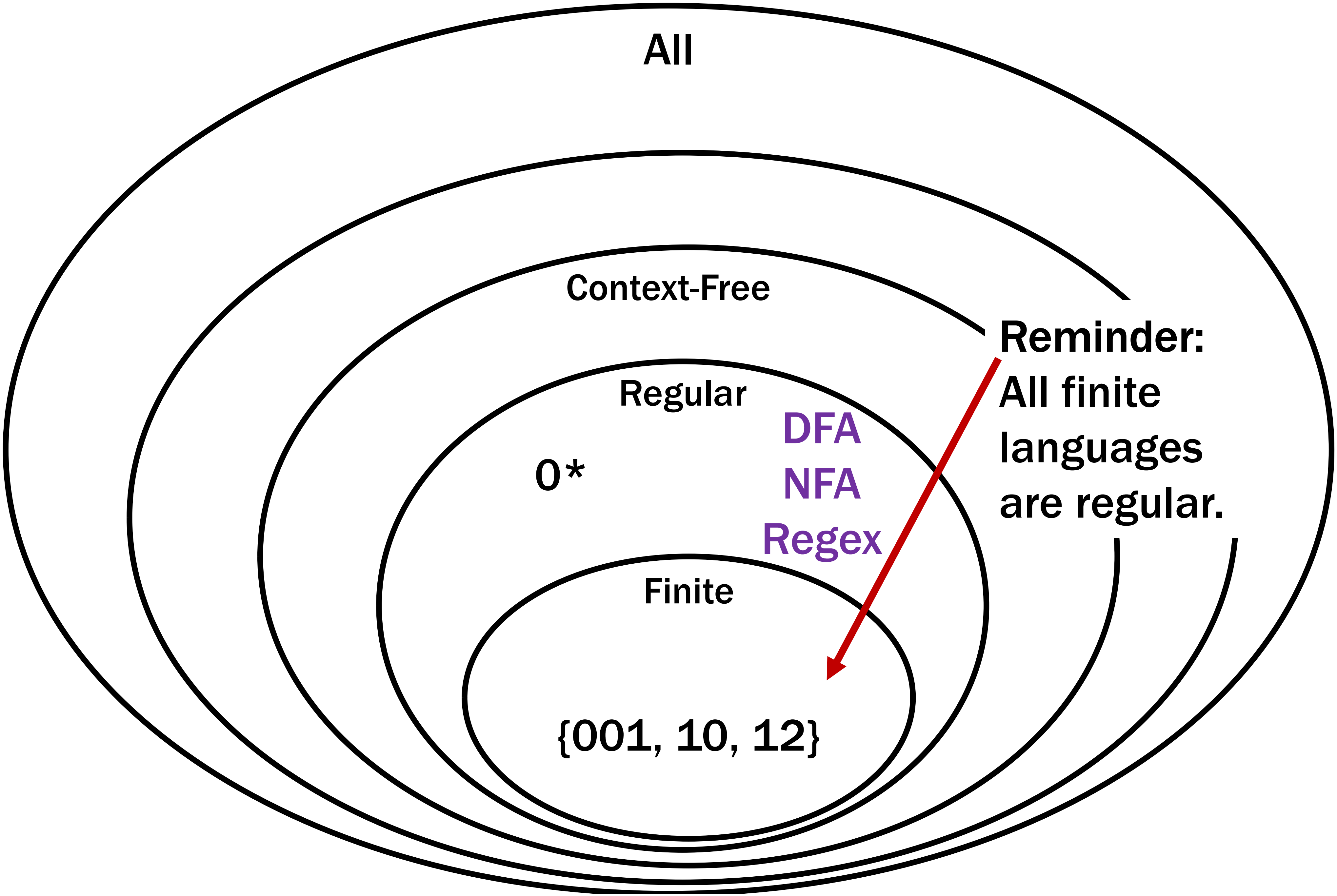
What languages have DFAs? CFGs?

All of them?

Languages and Representations!



Languages and Representations!

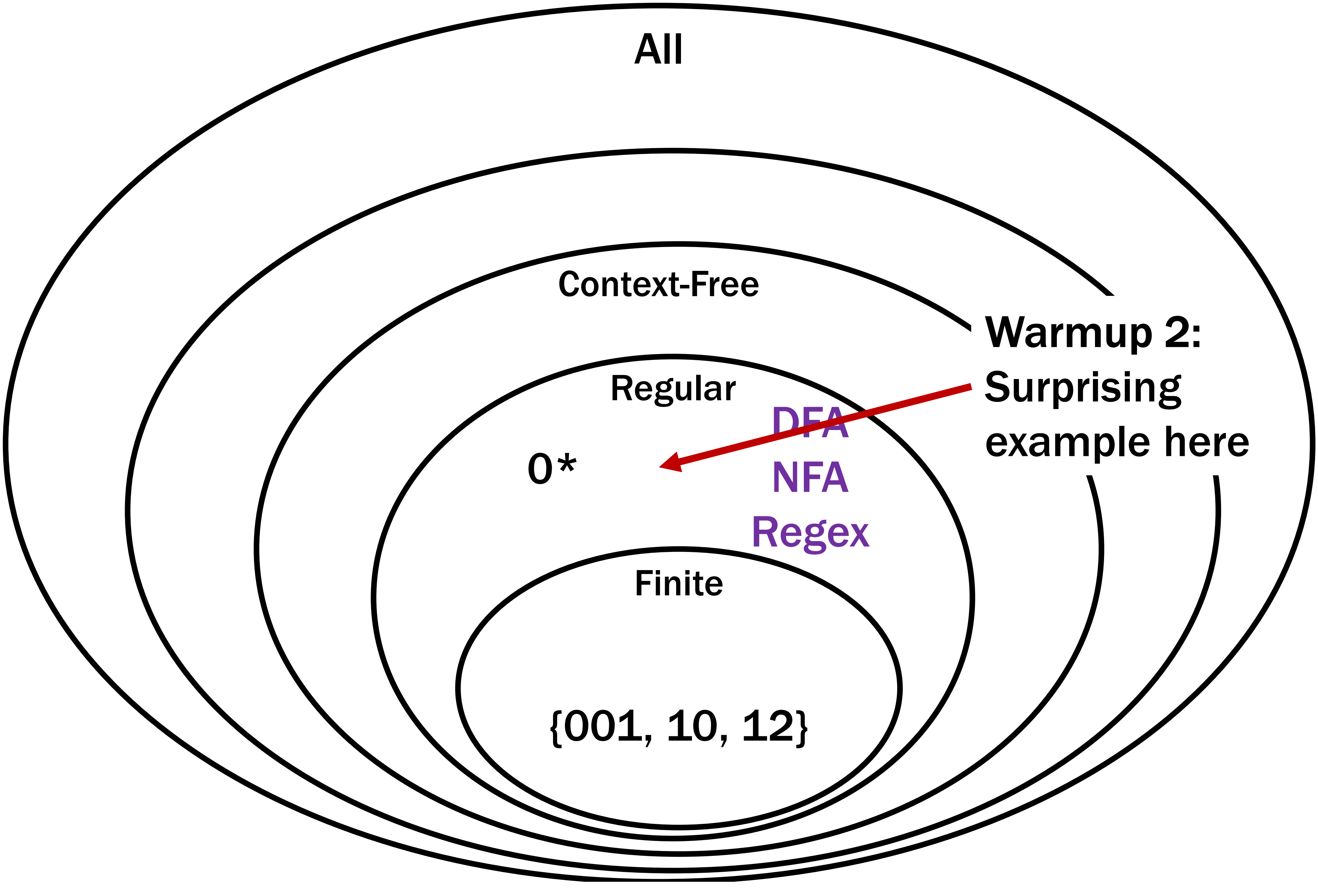


DFAs Recognize Any Finite Language

Construct a DFA for each string in the language.

Then, put them together using the union construction.

Languages and Machines!



An Interesting Infinite Regular Language

L = {x∈ {0, 1}*: x has an equal number of substrings 01 and 10}.

L is infinite.

0, 00, 000, …

L is regular. How could this be?

That seems to require comparing counts…

- easy for a CFG
- but seems hard for DFAs!

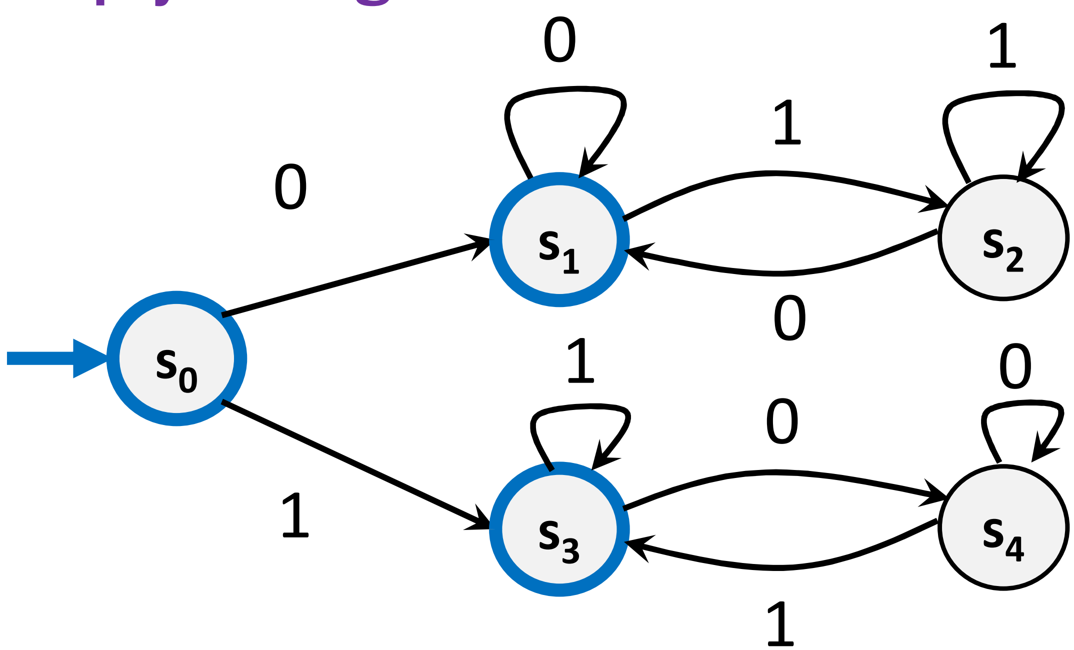
An Interesting Infinite Regular Language

L = {x∈ {0, 1}*: x has an equal number of substrings 01 and 10}.

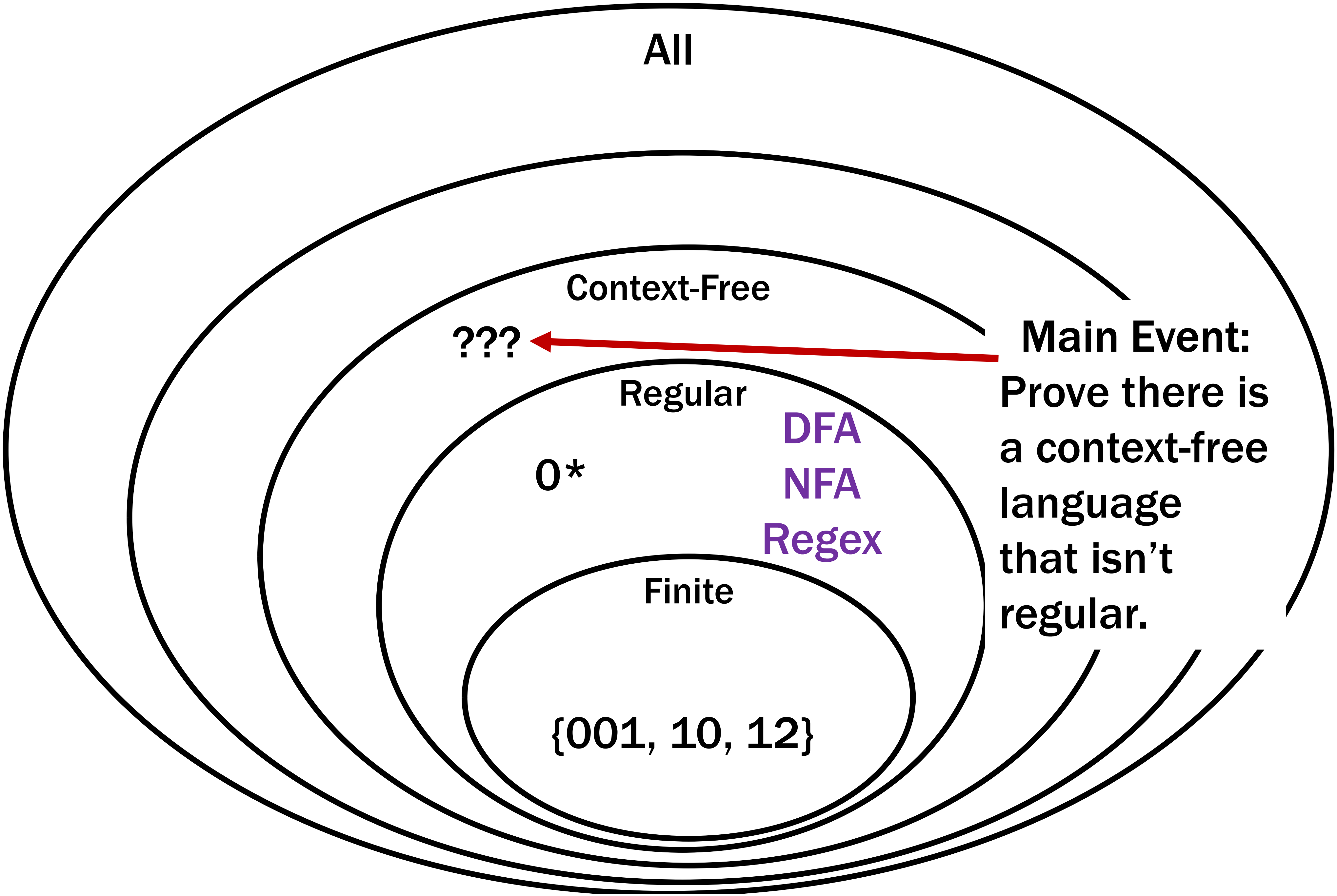
L is infinite.

0, 00, 000, ...

L is regular. How could this be? It is just the set of binary strings that are empty or begin and end with the same character!



Languages and Representations!

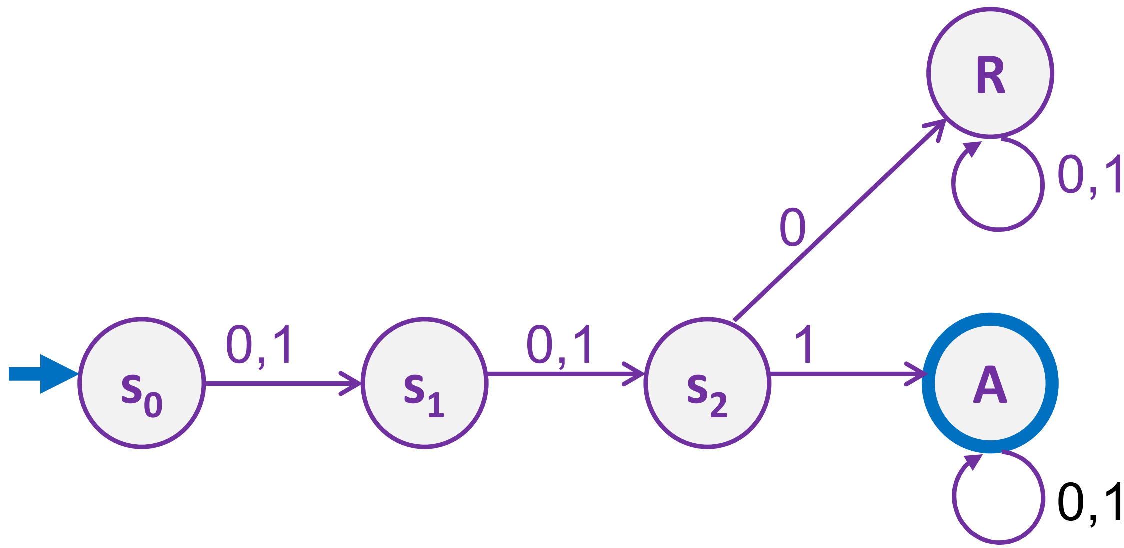


Tangent: How to prove a DFA minimal?

- Show there is no smaller DFA…

- Find a set of strings that *must* be distinguished
  - Such a set is a lower bound on the DFA size

Recall: Binary strings with a 1 in the 3rd position from the start



Distinguishing set:

{ε, 0, 00, 000, 001}

The language of “Binary Palindromes” is Context-Free

$$S \rightarrow \varepsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1$$

Is the language of "Binary Palindromes" Regular ?

Intuition (NOT A PROOF!):

Q: What would a DFA need to keep track of to decide?

A: It would need to keep track of the "first part" of the input in order to check the second part against it

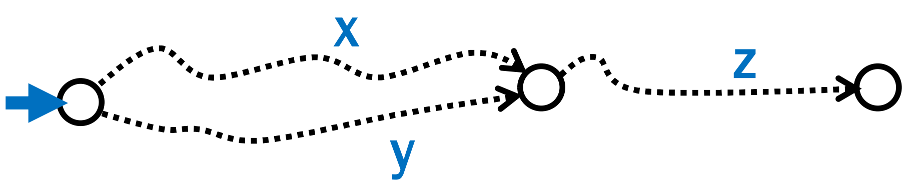
…but there are an infinite # of possible first parts and we only have finitely many states.

Proof idea: any machine that does not remember the entire first half will be wrong for some inputs

Useful Lemmas about DFAs

Lemma 1: If DFA **M** takes $\mathbf{x}, \mathbf{y} \in \boldsymbol{\Sigma}^*$ to the same state, then for every $\mathbf{z} \in \boldsymbol{\Sigma}^*$, M accepts $\mathbf{x} \bullet \mathbf{z}$ iff it accepts $\mathbf{y} \bullet \mathbf{z}$.

M can't remember that the input was $\mathbf{x}$, not $\mathbf{y}$.



$$x \bullet z \ = \ x_1\, x_2 \ldots x_n\, z_1\, z_2 \ldots z_k$$

$$y \bullet z \ = \ y_1\, y_2 \ldots y_m\, z_1\, z_2 \ldots z_k$$

Useful Lemmas about DFAs

Lemma 2: If DFA **M** has **n** states and a set **S** contains *more* than **n** strings, then **M** takes at least two strings from **S** to the same state.

M can't take n+1 or more strings to different states because it doesn't have n+1 different states.

So, some pair of strings must go to the same state.

B = {binary palindromes} can't be recognized by any DFA

Suppose for contradiction that some DFA, M, recognizes B.

We will show M accepts or rejects a string it shouldn't.

Consider $S = \{1, 01, 001, 0001, 00001, \ldots\} = \{0^n1 : n \geq 0\}$.

B = {binary palindromes} can't be recognized by any DFA

Suppose for contradiction that some DFA, M, accepts B.

We will show M accepts or rejects a string it shouldn't.

Consider $S = \{1, 01, 001, 0001, 00001, \ldots\} = \{0^n 1 : n \geq 0\}$.

Since there are finitely many states in M and infinitely many strings in S, by Lemma 2, there exist strings $0^a 1 \in S$ and $0^b 1 \in S$ with $a \neq b$ that end in the same state of M.

SUPER IMPORTANT POINT: You do not get to choose what a and b are. Remember, we've just proven they exist…we must take the ones we're given!

B = {binary palindromes} can't be recognized by any DFA

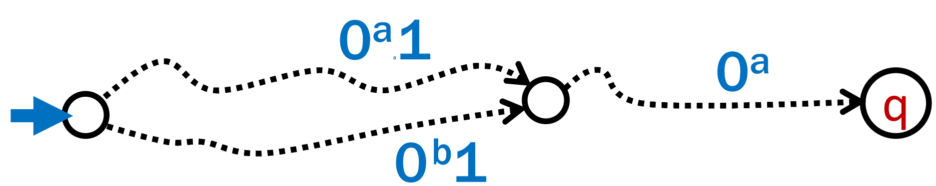
Suppose for contradiction that some DFA, M, accepts B.

We will show M accepts or rejects a string it shouldn't.

Consider $S = \{1, 01, 001, 0001, 00001, \ldots\} = \{0^n1 : n \geq 0\}$.

Since there are finitely many states in M and infinitely many strings in S, by Lemma 2, there exist strings $0^a1 \in S$ and $0^b1 \in S$ with $a \neq b$ that end in the same state of M.

Now, consider appending 0^a to both strings.



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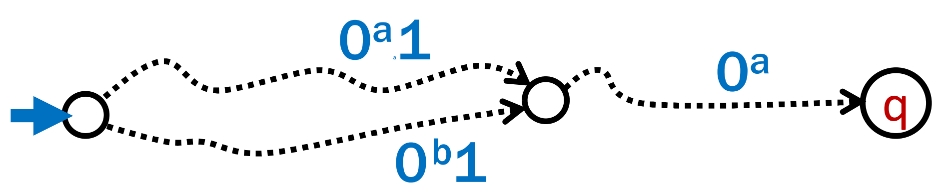

Since 0^a1 and 0^b1 end in the same state, 0^a10^a and 0^b10^a also end in the same state, call it q. But then M makes a mistake: q needs to be an accept state since $0^a10^a \in$ B, but M would accept $0^b10^a \notin$ B, which is an error.

B = {binary palindromes} can't be recognized by any DFA

Suppose for contradiction that some DFA, M, accepts B.

We will show M accepts or rejects a string it shouldn't.

Consider $S = \{1, 01, 001, 0001, 00001, \ldots\} = \{0^n1 : n \geq 0\}$.

Since there are finitely many states in M and infinitely many strings in S, by Lemma 2, there exist strings $0^a1 \in S$ and $0^b1 \in S$ with $a \neq b$ that end in the same state of M.

Now, consider appending 0^a to both strings.

Since 0^a1 and 0^b1 end in the same state, 0^a10^a and 0^b10^a also end in the same state, call it q. But then M makes a mistake: q needs to be an accept state since $0^a10^a \in B$, but M would accept $0^b10^a \notin B$, which is an error.

This proves that M does not recognize B, contradicting our assumption that it does. Thus, no DFA recognizes B.

Showing that a Language L is not regular

1. "Suppose for contradiction that some DFA M recognizes L."
2. Consider an INFINITE set S of prefixes (which we intend to complete later).
3. "Since S is infinite and M has finitely many states, there must be two strings s_a and s_b in S for $s_a \neq s_b$ that end up at the same state of M."
4. Consider appending the (correct) completion t to each of the two strings.
5. "Since s_a and s_b both end up at the same state of M, and we appended the same string t, both $s_a t$ and $s_b t$ end at the same state q of M. Since $s_a t \in L$ and $s_b t \notin L$, M does not recognize L."
6. "Thus, no DFA recognizes L."

Showing that a Language L is not regular

The choice of S is the creative part of the proof

You must find an infinite set S with the property that *no two* strings can be taken to the same state

- i.e., for *every pair* of strings S there is an “accept” completion that the two strings DO NOT SHARE

Prove A = $\{0^n1^n : n \geq 0\}$ is not regular

Suppose for contradiction that some DFA, M, recognizes A.

Let S =

Prove $A = \{0^n1^n : n \geq 0\}$ is not regular

Suppose for contradiction that some DFA, M, recognizes A.

Let $S = \{0^n : n \geq 0\}$. Since S is infinite and M has finitely many states, there must be two strings, 0^a and 0^b for some $a \neq b$ that end in the same state in M.

Prove **A** = $\{0^n1^n : n \geq 0\}$ is not regular

Suppose for contradiction that some DFA, M, recognizes A.

Let S = $\{0^n : n \geq 0\}$**. Since S is infinite and M has finitely many states, there must be two strings,** 0^a **and** 0^b **for some** $a \neq b$ **that end in the same state in M.**

Consider appending 1^a **to both strings.**

Prove A = $\{0^n1^n : n \geq 0\}$ is not regular

Suppose for contradiction that some DFA, **M**, recognizes **A**.

Let **S** = $\{0^n : n \geq 0\}$. Since **S** is infinite and **M** has finitely many states, there must be two strings, 0^a and 0^b for some $a \neq b$ that end in the same state in **M**.

Consider appending 1^a to both strings.

Note that $0^a1^a \in$ **A**, but $0^b1^a \notin$ **A** since $a \neq b$. But they both end up in the same state of **M**, call it **q**. Since $0^a1^a \in$ **A**, state **q** must be an accept state but then **M** would incorrectly accept $0^b1^a \notin$ **A** so **M** does not recognize **A**.

Thus, no DFA recognizes **A**.

Prove P = {balanced parentheses} is not regular

Suppose for contradiction that some DFA, M, accepts P.

Let S =

Prove P = {balanced parentheses} is not regular

Suppose for contradiction that some DFA, M, recognizes P.

Let S = $\{\, (^n : n \geq 0\}$**. Since S is infinite and M has finitely many states, there must be two strings,** $(^a$ **and** $(^b$ **for some** $a \neq b$ **that end in the same state in M.**

Prove **P** = {balanced parentheses} is not regular

Suppose for contradiction that some DFA, **M**, recognizes **P**.

Let **S** = $\{ (^n : n \geq 0\}$. Since **S** is infinite and **M** has finitely many states, there must be two strings, $(^a$ and $(^b$ for some $a \neq b$ that end in the same state in **M**.

Consider appending $)^a$ to both strings.

Prove P = {balanced parentheses} **is not regular**

Suppose for contradiction that some DFA, M, recognizes P.

Let $S = \{ (^n : n \geq 0\}$**. Since** S **is infinite and** M **has finitely many states, there must be two strings,** $(^a$ **and** $(^b$ **for some** $a \neq b$ **that end in the same state in** M**.**

Consider appending $)^a$ **to both strings.**

Note that $(^a)^a \in P$**, but** $(^b)^a \notin P$ **since** $a \neq b$**. But they both end up in the same state of** M**, call it** q**. Since** $(^a)^a \in P$**, state** q **must be an accept state but then** M **would incorrectly accept** $(^b)^a \notin P$ **so** M **does not recognize** P**.**

Thus, no DFA recognizes P**.**

Showing that a Language L is not regular

1. "Suppose for contradiction that some DFA M recognizes L."
2. Consider an INFINITE set S of prefixes (which we intend to complete later). It is imperative that for *every pair* of strings in our set there is an "accept" completion that the two strings DO NOT SHARE.
3. "Since S is infinite and M has finitely many states, there must be two strings s_a and s_b in S for $s_a \neq s_b$ that end up at the same state of M."
4. Consider appending the (correct) completion t to each of the two strings.
5. "Since s_a and s_b both end up at the same state of M, and we appended the same string t, both s_at and s_bt end at the same state q of M. Since $s_at \in L$ and $s_bt \notin L$, M does not recognize L."
6. "Thus, no DFA recognizes L."

Fact: This method is optimal

- Suppose that for a language L, the set S is a *largest* set of prefixes with the property that, for every pair $s_a \neq s_b \in S$, there is some string t such that one of $s_a t$, $s_b t$ is in L but the other isn't.
- If S is infinite, then L is not regular
- If S is finite, then the minimal DFA for L has precisely $|S|$ states, one reached by each member of S.

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- If S is infinite, then L is not regular
- If S is finite, then the minimal DFA for L has precisely $|S|$ states, one reached by each member of S.

Corollary: Our minimization algorithm was correct.

– we separated *exactly* those states for which some t would make one accept and another not accept

Important Notes

- It is not necessary for our strings xz with $x \in L$ to allow any string in the language
  - we only need to find a small "core" set of strings that must be distinguished by the machine

- It is not true that, if L is irregular and $L \subseteq U$, then U is irregular!
  - we always have $L \subseteq \Sigma^*$ and Σ^* is regular!
  - our argument needs different answers: $xz \in L \not\leftrightarrow yz \in L$

    for Σ^*, both strings are always in the language

Do not claim in your proof that, because $L \subseteq U$, U is also irregular

New Machinery: Generalized NFAs

- Like NFAs but allow
  - parallel edges (between the same pair of states)
  - regular expressions as edge labels

    NFAs already have edges labeled ε or ***a***

- Machine can follow an edge labeled by A by reading a string of input characters in the language of A
  - (if A is *a* or ε, this matches the original definition, but we now allow REs built with recursive steps.)

New Machinery: Generalized NFAs

- Like NFAs but allow
  - parallel edges
  - regular expressions as edge labels

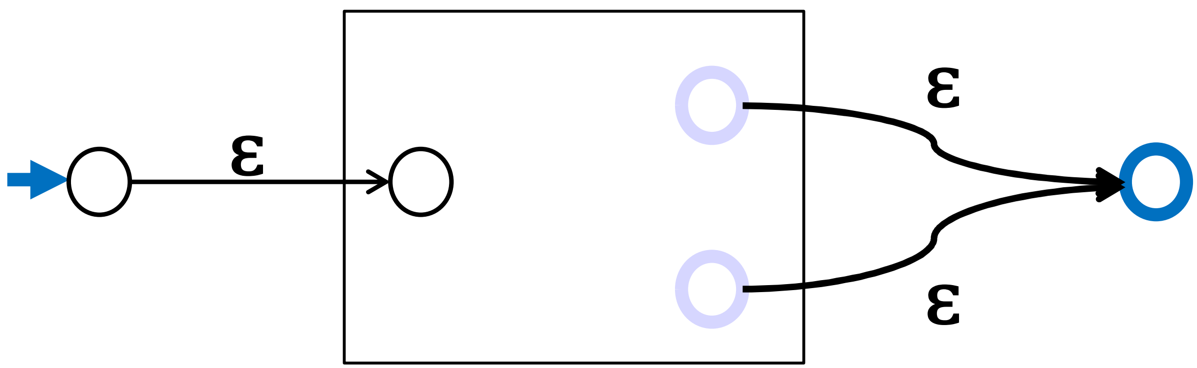
    NFAs already have edges labeled ε or ***a***

- The label of a path is now the concatenation of the regular expressions on those edges, making it a regular expression

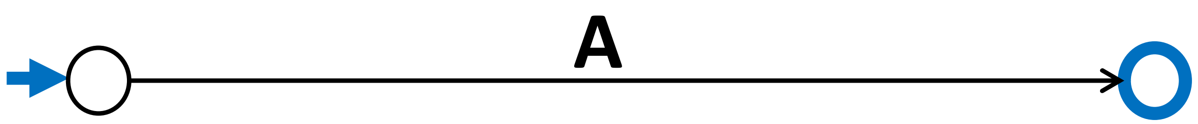
- Def: A string x is accepted by a generalized NFA iff there is a *path* from start to final state labeled by a regular expression whose language contains x

Construction Idea

Add new start state and final state

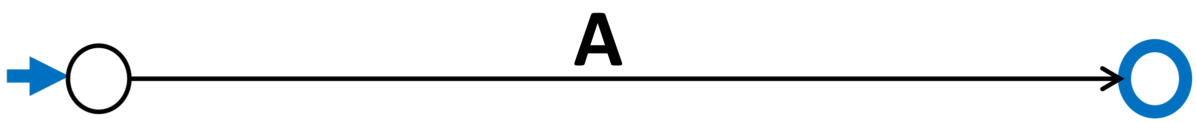


Then delete the original states one by one,
adding edges to keep the same language,
until the graph looks like:



Starting from an NFA

Then delete the original states one by one,
adding edges to keep the same language,
until the graph looks like:

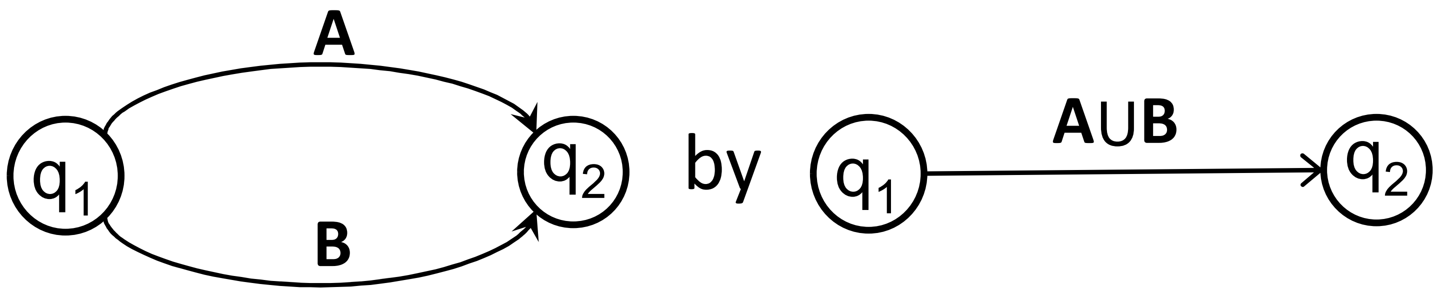


Final graph has only one path to the accepting state,
which is labeled by A,
so it accepts iff x is in the language of A

Thus, A is a regular expression with the same language as the original NFA.

Only two simplification rules

- **Rule 1**: For any two states q_1 and q_2 with parallel edges (possibly q_1=q_2), replace

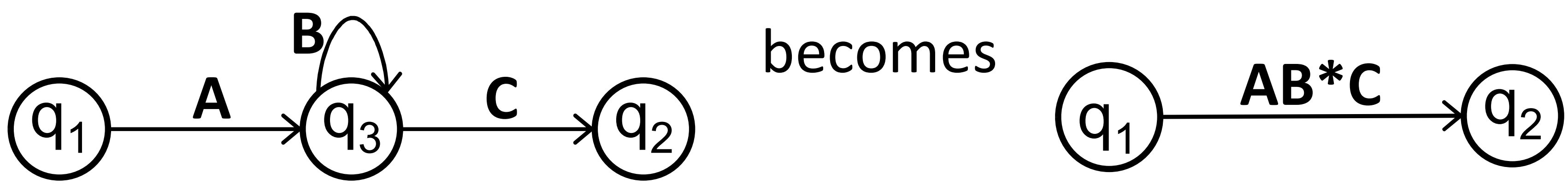


If the machine would have used the edge labeled A by consuming an input x in the language of A, it can instead use the edge labeled A∪B.

Furthermore, this new edge does not allow transitions for any strings other than those that matched A or B.

Only two simplification rules

- Rule 2: Eliminate non-start/accepting state q_3 by creating direct edges that skip q_3

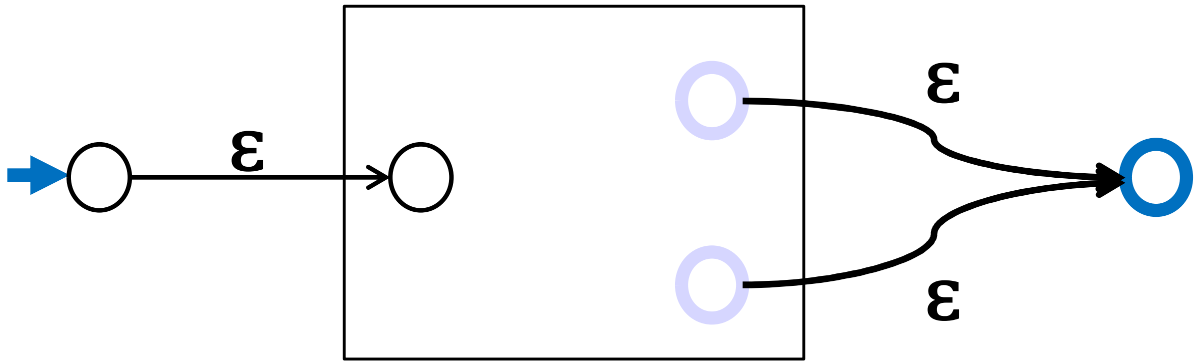


for *every* pair of states q_1, q_2 (even if q_1=q_2)

Any path from q_1 to q_2 would have to match AB^nC for some n (the number of times the self loop was used), so the machine can use the new edge instead. New edge *only* allows strings that were allowed before.

Construction Overview

Add new start state and final state



While the box contains some state s:

for all states r, t with (r, s) and (s, t) in E:

create a direct edge (r, t) by Rule 2

delete s (no longer needed)

merge all parallel edges by Rule 1

Construction Overview

While the box contains some state s:

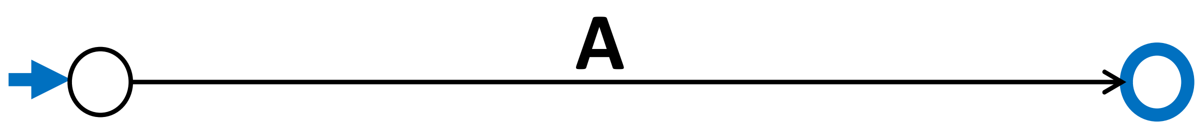
for all states r, t with (r, s) and (s, t) in E:

create a direct edge (r, t) by Rule 2

delete s (no longer needed)

merge all parallel edges by Rule 1

When the loop exits, the graph looks like this:

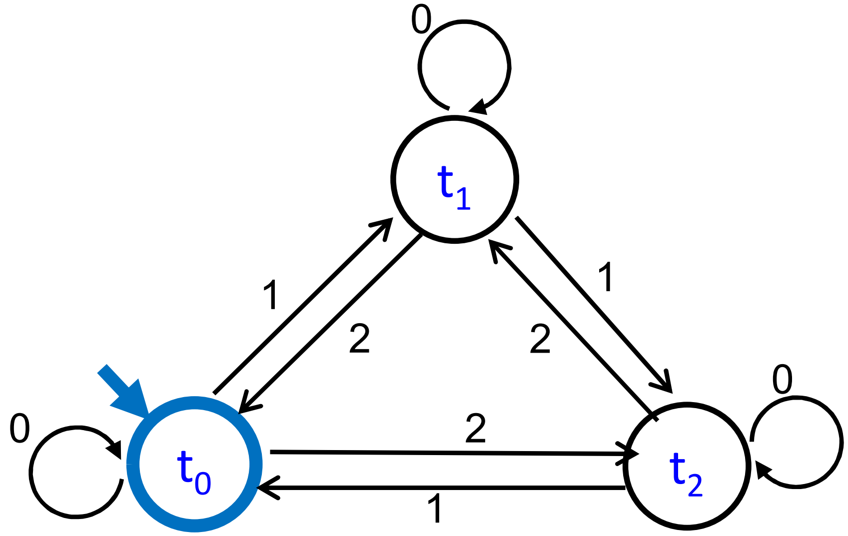


A is a regular expression with the same language as the original NFA.

Converting an NFA to a regular expression

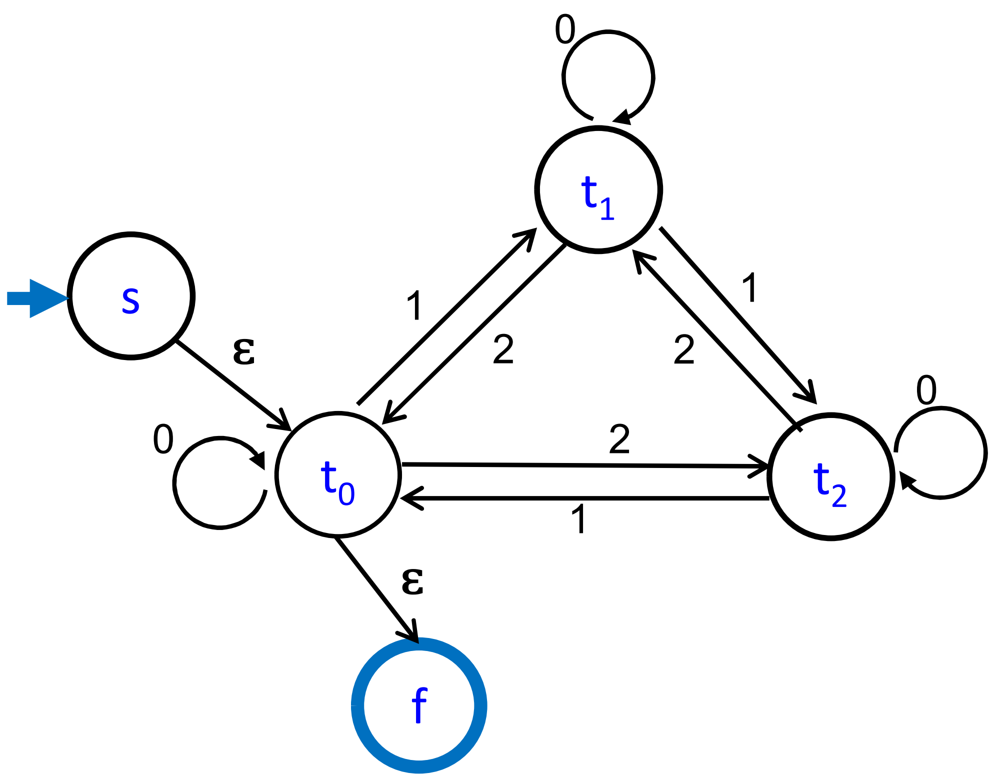
Consider the DFA for the mod 3 sum

- Accept strings from {0,1,2}* where the digits mod 3 sum of the digits is 0



Splicing out a state t_1

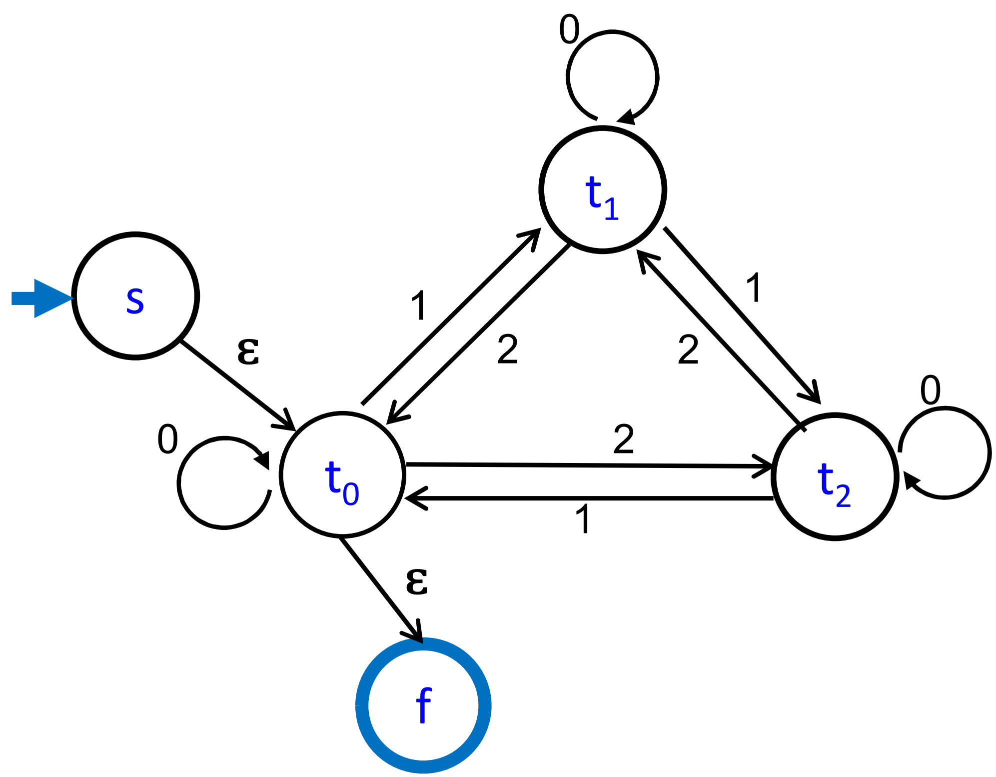
Create direct edges between neighbors of t_1 (so that we can delete it afterward)



Splicing out a state t_1

Regular expressions to add to edges

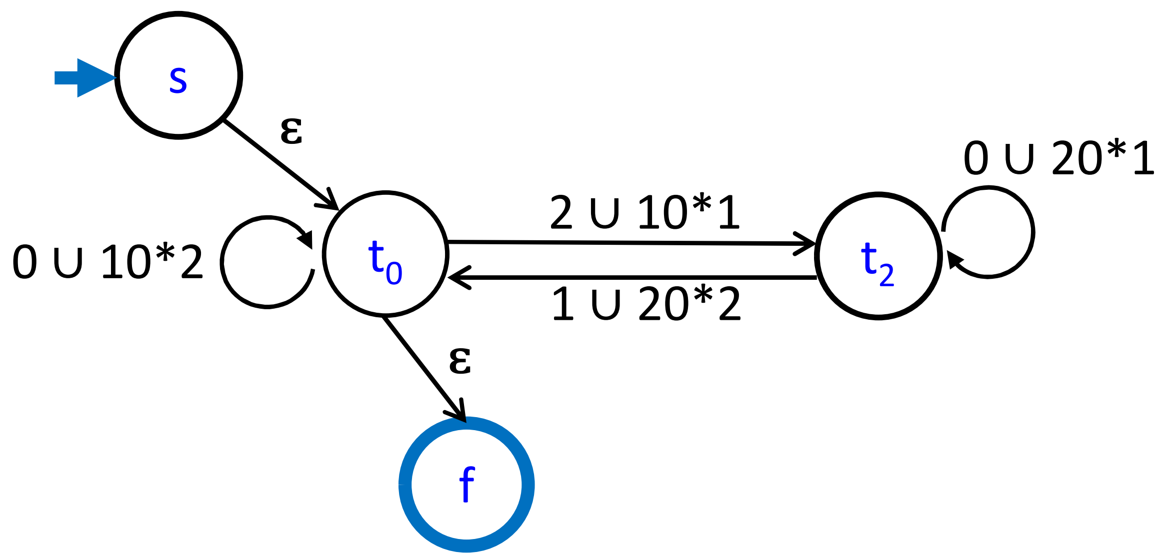
$t_0 \to t_1 \to t_0$: 10*2
$t_0 \to t_1 \to t_2$: 10*1
$t_2 \to t_1 \to t_0$: 20*2
$t_2 \to t_1 \to t_2$: 20*1



Splicing out a state t_1

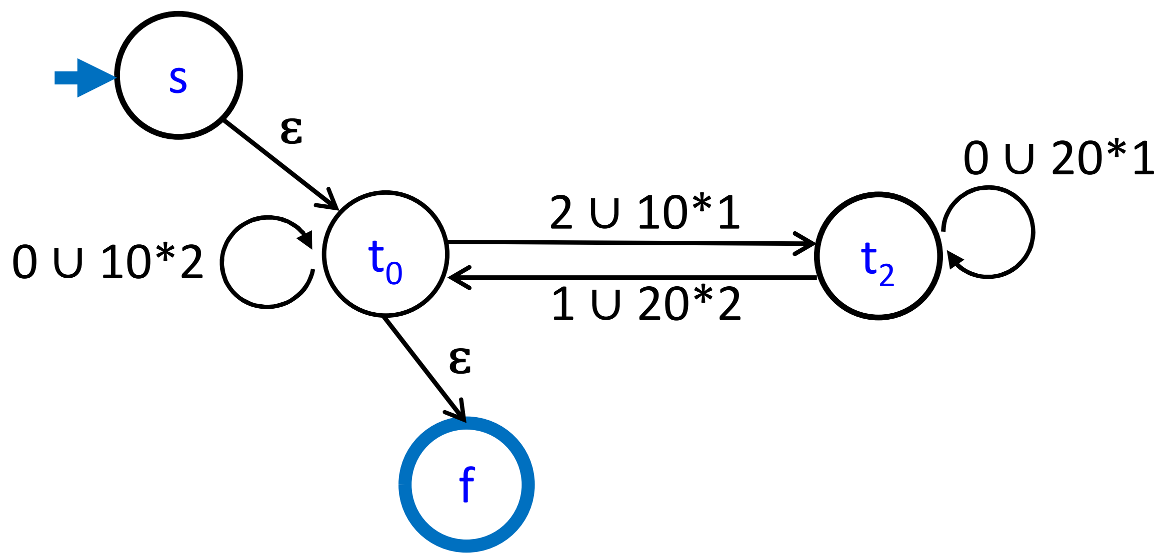
Delete t_1 now that it is redundant

$t_0 \to t_1 \to t_0$: 10*2
$t_0 \to t_1 \to t_2$: 10*1
$t_2 \to t_1 \to t_0$: 20*2
$t_2 \to t_1 \to t_2$: 20*1



Splicing out a state t_1

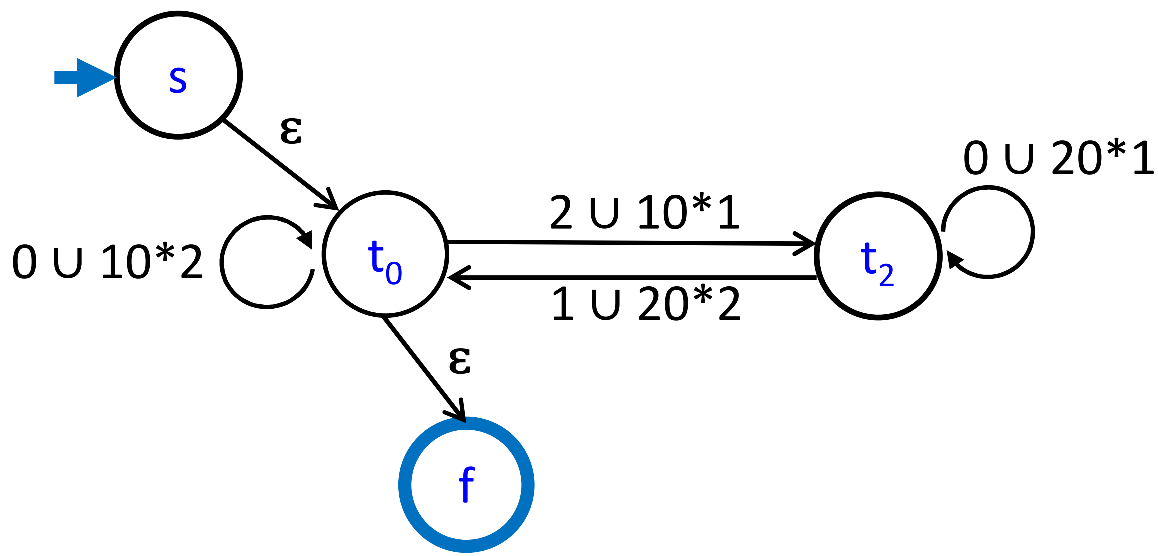
Create direct edges between neighbors of t_2 (so that we can delete it afterward)



Splicing out a state t_1

Regular expressions to add to edges

R_1: $0 \cup 10^*2$
R_2: $2 \cup 10^*1$
R_3: $1 \cup 20^*2$
R_4: $0 \cup 20^*1$



Splicing out state t_2 (and then t_0)

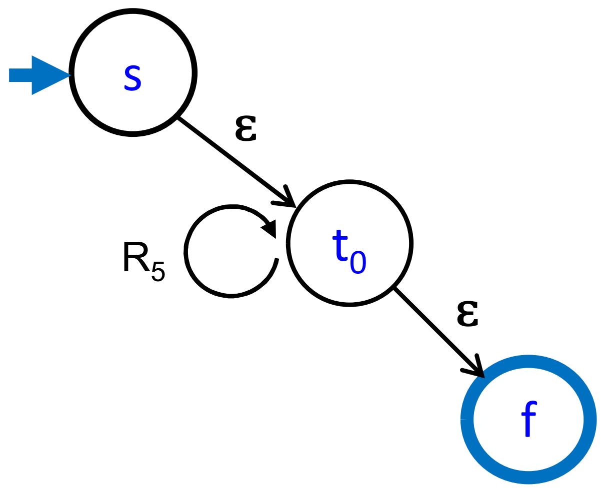
Delete t_2 now that it is redundant

R_1: $0 \cup 10^*2$

R_2: $2 \cup 10^*1$

R_3: $1 \cup 20^*2$

R_4: $0 \cup 20^*1$

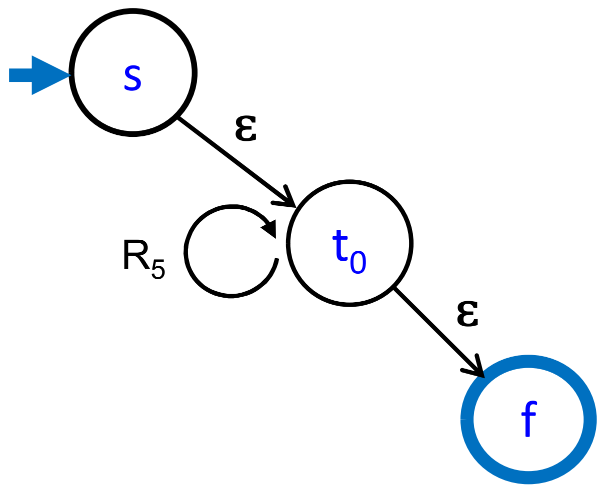


R_5: $R_1 \cup R_2R_4^*R_3$

Splicing out state t_2 (and then t_0)

Create direct (s,f) edge so we can delete t_0

R_1: $0 \cup 10^*2$
R_2: $2 \cup 10^*1$
R_3: $1 \cup 20^*2$
R_4: $0 \cup 20^*1$
R_5: $R_1 \cup R_2R_4{}^*R_3$

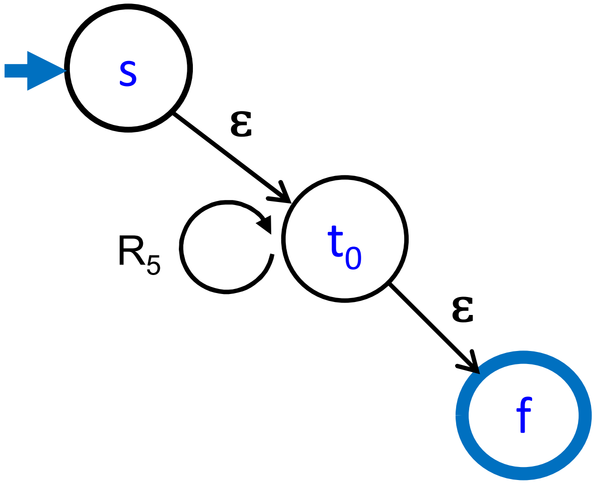


Splicing out state t_2 (and then t_0)

Regular expressions to add to edges

R_1: $0 \cup 10^*2$
R_2: $2 \cup 10^*1$
R_3: $1 \cup 20^*2$
R_4: $0 \cup 20^*1$
R_5: $R_1 \cup R_2R_4{}^*R_3$

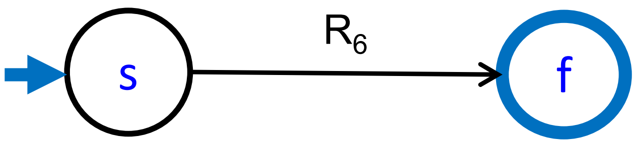
$t_0 \rightarrow t_1 \rightarrow t_0$: $R_5{}^*$



Splicing out state t_2 (and then t_0)

Delete t_0 now that it is redundant

R_1: $0 \cup 10^*2$
R_2: $2 \cup 10^*1$
R_3: $1 \cup 20^*2$
R_4: $0 \cup 20^*1$
R_5: $R_1 \cup R_2R_4{}^*R_3$

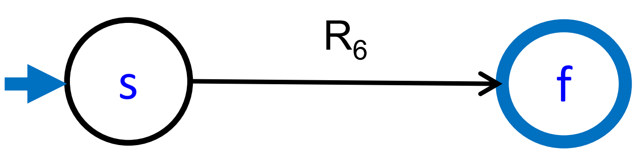


R_6: $R_5{}^*$

Splicing out state t_2 (and then t_0)

Regular expressions to add to edges

R_1: $0 \cup 10^*2$
R_2: $2 \cup 10^*1$
R_3: $1 \cup 20^*2$
R_4: $0 \cup 20^*1$
R_5: $R_1 \cup R_2R_4{}^*R_3$
R_6: $R_5{}^*$



Final regular expression: R_6 =
$(0 \cup 10^*2 \cup (2 \cup 10^*1)(0 \cup 20^*1)^*(1 \cup 20^*2))^*$

Application of FSMs: Pattern matching

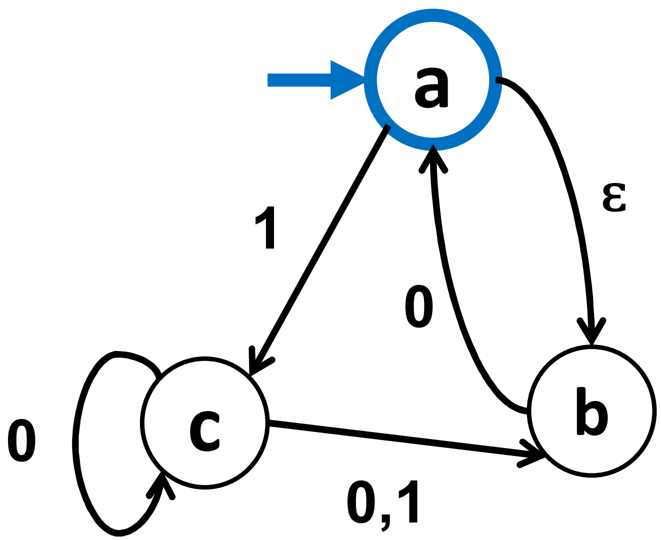
- Given
  - a string s of n characters
  - a pattern p of m characters
  - usually $m \ll n$
- Find
  - all occurrences of the pattern p in the string s

- Obvious algorithm:
  - try to see if p matches at each of the positions in s stop at a failed match and try matching at the next position: $O(mn)$ running time.

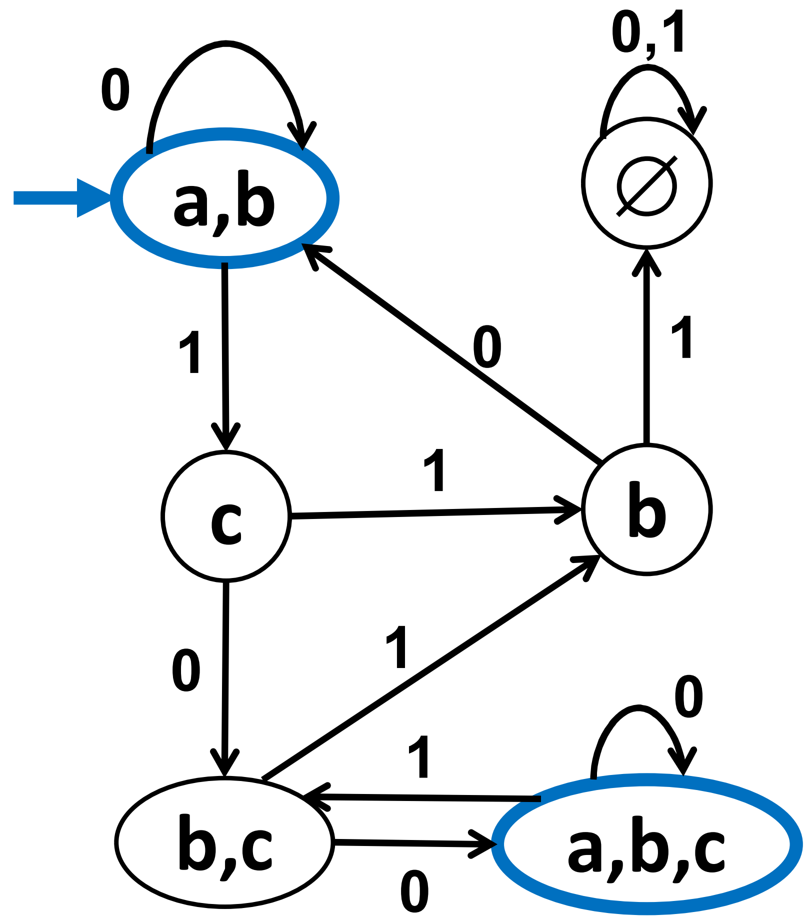
Application of FSMs: Pattern Matching

- With DFAs can do this in $O(m+n)$ time.
- See Extra Credit problem on HW8 for some ideas of how to get to $O(m^2 + n)$.

Last time: NFA to DFA



NFA



DFA

Exponential Blow-up in Simulating Nondeterminism

- In general the DFA might need a state for every subset of states of the NFA
  - Power set of the set of states of the NFA
  - n-state NFA yields DFA with at most 2^n states
  - We saw an example where roughly 2^n is necessary
    "Is the n^{th} char from the end a 1?"

The famous "P=NP?" question asks whether a similar blow-up is always necessary to get rid of nondeterminism for polynomial-time algorithms