Strong Induction

CSE 311: Foundations of Computing I
Lecture 14

Announcements

- HW4 due Friday at 11:59 pm
  - There are 2 submission spots on Gradescope:
    Feedback before the midterm is only guaranteed if you don't use late days
- Midterm this Monday in class
- Homework 5 releases on Monday

Midterm

- The reference sheets will be provided
- One practice midterm and solutions are posted
- Optional review session this Saturday, July 20th at 11:00am

Strong Induction

Let’s Try Another Induction Proof

Fundamental Theorem of Arithmetic

Every positive integer greater than 1 has a unique prime factorization.

Uniqueness is hard. Let’s just show existence.

I.e.

Claim: Every positive integer greater than 1 can be written as a product of primes.

Prime Factorizations

Some examples

$$12 = 2^2 \cdot 3$$
$$35 = 5 \cdot 7$$
$$36 = 2^2 \cdot 3^2$$
$$7 = 7$$

Notice, for prime numbers the product is just the one number.

Induction on Primes.

Let $P(n)$ be "n can be written as a product of primes."

We show $P(n)$ for all integers $n \geq 2$ by induction on n.

Base Case ($\boldsymbol{n = 2}$): 2 is a product of just itself. Since 2 is prime, it is written as a product of primes.

Inductive Hypothesis:

Inductive Step:
Case 1, $k + 1$ is prime:

Case 2, $k + 1$ is composite:

Therefore $P(k + 1)$.

$P(n)$ holds for all $n \geq 2$ by the principle of induction.

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We show $P(n)$ for all integers $n \geq 2$ by induction on n.

Base Case ($\boldsymbol{n = 2}$): 2 is a product of just itself. Since 2 is prime, it is written as a product of primes.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary integer $k \geq 2$.

Inductive Step:
Case 1, $k + 1$ is prime: then $k + 1$ is automatically written as a product of primes.

Case 2, $k + 1$ is composite:

Therefore $P(k + 1)$.

$P(n)$ holds for all $n \geq 2$ by the principle of induction.

We're Stuck

We can divide $k + 1$ up into smaller pieces (say s, t such that $st = k + 1$ with $2 \leq s < k + 1$ and $2 \leq t < k + 1$

Is $P(s)$ true? Is $P(t)$ true?

I mean…it would be…

But in the inductive step we don't have it…

Let's add it to our inductive hypothesis.

Recall: Induction

Induction relied on the fact that:

$$\forall n\, \mathrm{P}(n) \equiv \mathrm{P}(0) \land \forall k \left(\mathrm{P}(k) \to \mathrm{P}(k+1)\right)$$

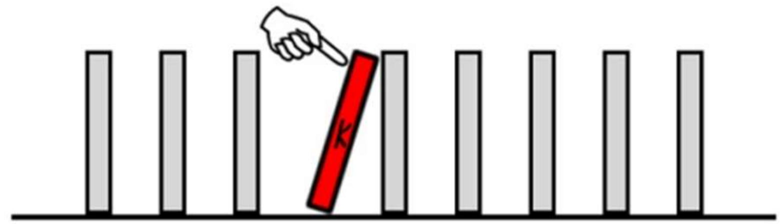
Base Case
Prove $\mathrm{P}(0)$ holds.

Inductive Hypothesis
Let $k \geq 0$ be an arbitrary integer.
Suppose $\mathrm{P}(k)$ holds.

Inductive Step
Prove that $\mathrm{P}(k+1)$ holds (using $\mathrm{P}(k)$)

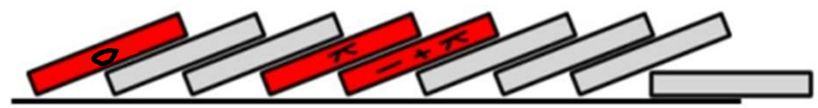
Recall: Induction

Check that the formula holds for $n = 0$



Assume the formula holds for $n = k$.

Show that the assumption *implies* that the formula holds for $n = k + 1$.



Conclude that the formula holds for all $n \in \mathbb{N}$.

Another Equivalence

There are other statements that are stronger but still useful to $\forall n\ \mathrm{P}(n)$.
In particular:

$$\forall n\ \mathrm{P}(n) \equiv \mathrm{P}(0) \land \mathrm{P}(1) \land \mathrm{P}(2) \land \mathrm{P}(3) \ldots$$

$$\equiv \mathrm{P}(0) \land \big(\mathrm{P}(0) \rightarrow \mathrm{P}(1)\big) \land \Big(\big(\mathrm{P}(0) \land P(1)\big) \rightarrow \mathrm{P}(2)\Big) \land$$

$$\Big(\big(\mathrm{P}(0) \land P(1) \land \mathrm{P}(2)\big) \rightarrow \mathrm{P}(3)\Big) \ldots$$

$$\equiv \mathrm{P}(0) \land \forall k\ \Big(\big(\mathrm{P}(0) \land \cdots \land \mathrm{P}(k)\big) \rightarrow \mathrm{P}(k+1)\Big)$$

The Principle of Strong Induction

$$\mathrm{P}(0) \land \forall k \left(\left(\mathrm{P}(0) \land \cdots \land \mathrm{P}(k) \right) \rightarrow \mathrm{P}(k+1) \right)$$

<u>Base Case</u>
Prove $\mathrm{P}(0)$ holds.

<u>Inductive Hypothesis</u>
Let $k \geq 0$ be an arbitrary integer. Suppose $\mathrm{P}(0) \land \cdots \land \mathrm{P}(k)$ hold.

<u>Inductive Step</u>
Prove that $\mathrm{P}(k+1)$ holds

Strong Induction

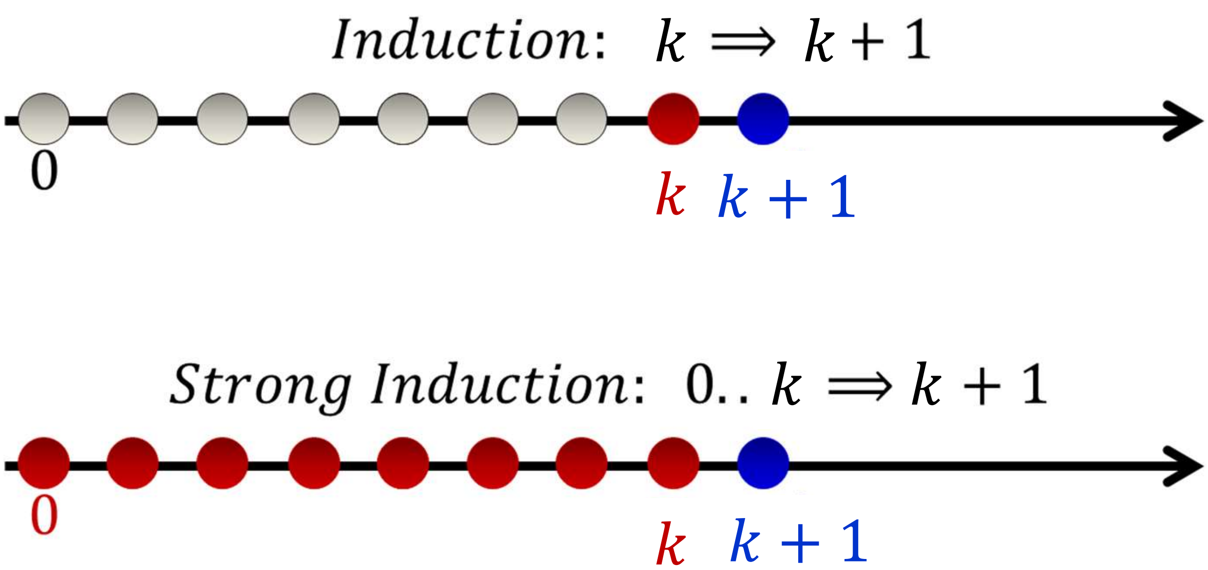
That hypothesis where we assume $P(\text{base case}), \ldots, P(k)$ instead of just $P(k)$ is called a strong inductive hypothesis.

Strong induction is the same fundamental idea as weak ("regular") induction.

$P(0)$ is true.
And $P(0) \rightarrow P(1)$, so $P(1)$.
And $P(1) \rightarrow P(2)$, so $P(2)$.
And $P(2) \rightarrow P(3)$, so $P(3)$.
And $P(3) \rightarrow P(4)$, so $P(4)$.
…

$P(0)$ is true.
And $P(0) \rightarrow P(1)$, so $P(1)$.
And $[\mathrm{P}(0) \land P(1)] \rightarrow P(2)$, so $P(2)$.
And $[\mathrm{P}(0) \land \cdots \land P(2)] \rightarrow P(3)$, so $P(3)$.
And$[\mathrm{P}(0) \land \cdots \land P(3)] \rightarrow P(4)$, so $P(4)$.
…

Strong Induction



Induction on Primes

Let $P(n)$ be "n can be written as a product of primes."

We show $P(n)$ for all integers $n \geq 2$ by induction on n.

Base Case ($\boldsymbol{n = 2}$): 2 is a product of just itself. Since 2 is prime, it is written as a product of primes.

Inductive Hypothesis:

Inductive Step:
Case 1, $k + 1$ is prime: then $k + 1$ is automatically written as a product of primes.

Case 2, $k + 1$ is composite:

Therefore $P(k + 1)$.

$P(n)$ holds for all $n \geq 2$ by the principle of induction.

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Base Case ($\boldsymbol{n = 2}$): 2 is a product of just itself. Since 2 is prime, it is written as a product of primes.

Inductive Hypothesis: Suppose $P(2), \ldots, P(k)$ hold for an arbitrary integer $k \geq 2$.

Inductive Step:
Case 1, $k + 1$ is prime: then $k + 1$ is automatically written as a product of primes.

Case 2, $k + 1$ is composite: We can write $k + 1 = st$ for s, t nontrivial divisors (i.e. $2 \leq s < k + 1$ and $2 \leq t < k + 1$). By inductive hypothesis, we can write s as a product of primes $p_1 \cdot \ldots p_j$ and t as a product of primes $q_1 \cdots q_\ell$. Multiplying these representations, $k + 1 = p_1 \cdots p_j \cdot q_1 \cdots q_\ell$, which is a product of primes.

Therefore $P(k + 1)$.

$P(n)$ holds for all $n \geq 2$ by the principle of induction.

Strong Induction vs. Weak Induction

- “Normal” Induction is otherwise known as Weak Induction
- All induction proofs could be written by Strong Induction instead. It’s a *stronger* hypothesis to use. There is more to work with.
- However, there’s often the philosophy to only use a stronger hypothesis when needed to make your inductive step more clear.

Making Induction Proofs Pretty

All of our strong induction proofs will come in 5 easy(?) steps!

1. Define $P(n)$. State that your proof is by induction on n.
2. Base Case: Show $P(b)$ i.e. show the base case
3. Inductive Hypothesis: Suppose $\mathrm{P(b)} \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq b$.
4. Inductive Step: Show $P(k+1)$ (i.e. get $[\mathrm{P(b)} \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$)
5. Conclude by saying $P(n)$ is true for all $n \geq b$ by the principle of induction.

Practical Advice

How many base cases do you need?

- Always at least one.
- If you're analyzing recursive code or a recursive function, at least one for each base case of the code/function.
- If you always go back s steps, at least s consecutive base cases.
- Enough to make sure every case is handled.

Strong Induction Example

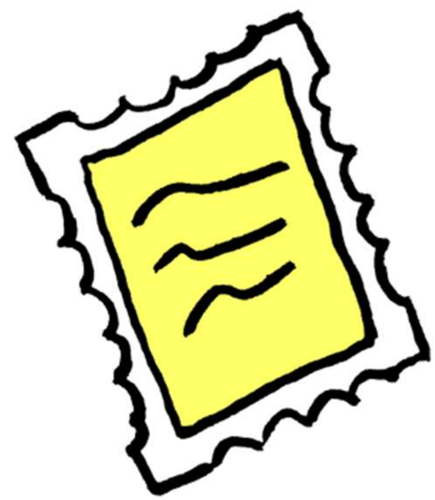
Stamp Collection

Stamp Collection

- I have a collection of $4¢$ and $5¢$ stamps. Prove that for all $n \geq 12$, I can make $n¢$ worth of stamps.

- Examples:
- $13¢ = 5¢ + 4¢ + 4¢$

- $22¢ = 5¢ + 5¢ + 4¢ + 4¢ + 4¢$



[Attempted Proof by Strong Induction]

Prove that for all $n \geq 12$, I can make n ¢ worth of stamps.

1. Let $\mathrm{P}(n)$ be "I can make n ¢ worth of stamps with just 4¢ and 5¢ stamps." We prove $\mathrm{P}(n)$ for all integers $n \geq 12$ by strong induction.
2. Base Case: 12¢ can be made with three 4¢ stamps. Thus $\mathrm{P}(12)$ is true.
3. IH: Suppose $\mathrm{P}(12) \wedge \cdots \wedge \mathrm{P}(k)$ hold for an arbitrary integer $k \geq 12$. I.e. we can make 12¢,13¢, $\ldots, k$¢ worth of stamps with just 4¢ and 5¢ stamps.
4. IS:

5. Conclusion: Thus $\mathrm{P}(n)$ holds for all integers $n \geq 12$ by strong induction.

[Attempted Proof by Strong Induction]

Prove that for all $n \geq 12$, I can make n ¢ worth of stamps.

1. Let $\mathrm{P}(n)$ be "I can make n ¢ worth of stamps with just 4¢ and 5¢ stamps." We prove $\mathrm{P}(n)$ for all integers $n \geq 12$ by strong induction.
2. Base Case: 12¢ can be made with three 4¢ stamps. Thus $\mathrm{P}(12)$ is true.
3. IH: Suppose $\mathrm{P}(12) \wedge \cdots \wedge \mathrm{P}(k)$ hold for an arbitrary integer $k \geq 12$. I.e. we can make 12¢,13¢, $\ldots, k$¢ worth of stamps with just 4¢ and 5¢ stamps.
4. IS: We aim to show $\mathrm{P}(k+1)$, i.e. that we can make $k+1$ cents in stamps. By the IH, we can make $k-3$ cents in stamps. Adding another 4¢ stamp gives exactly $k+1$ cents.
5. Conclusion: Thus $\mathrm{P}(n)$ holds for all integers $n \geq 12$ by strong induction.

WRONG

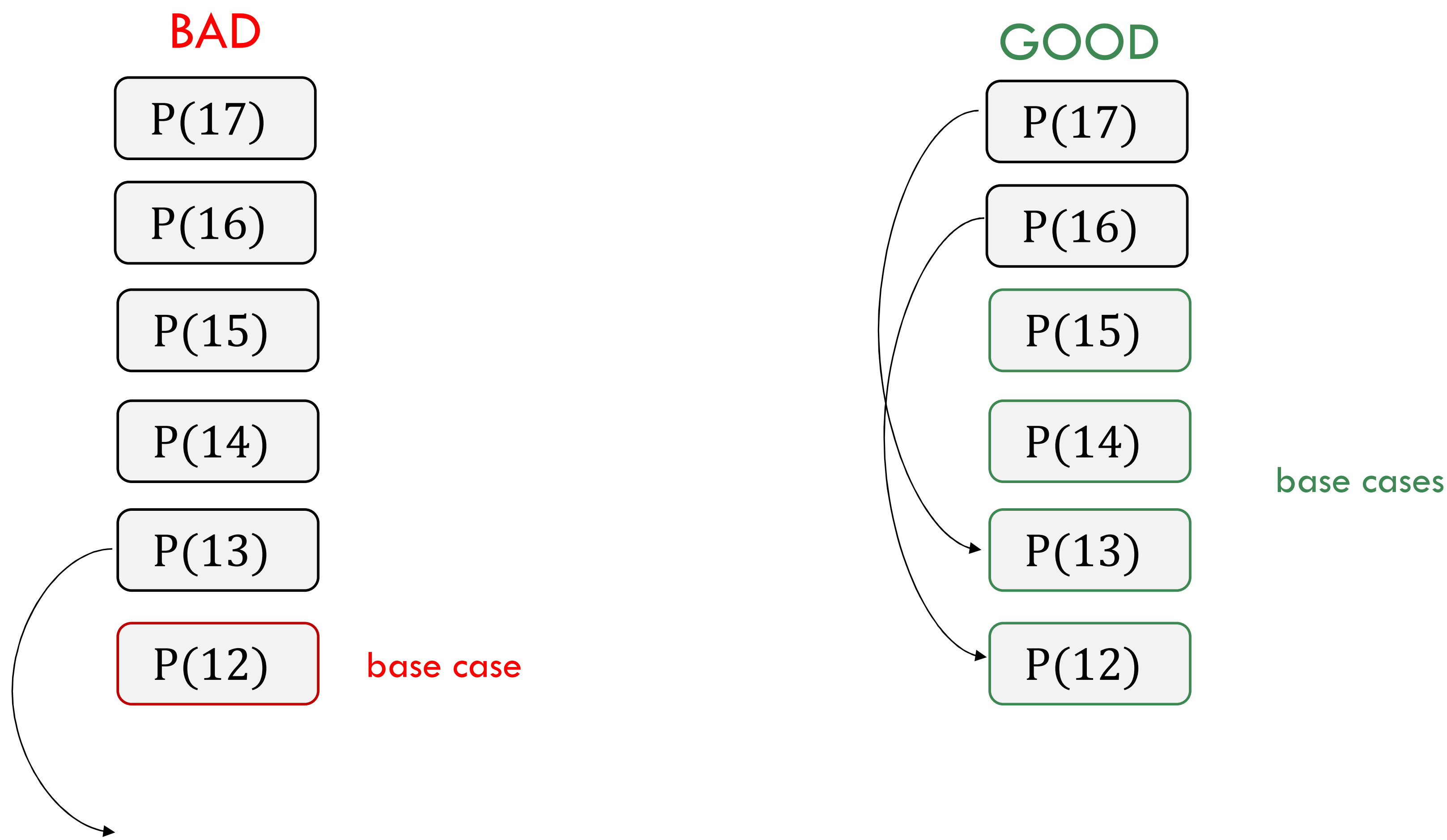
What was the problem?

- We don't know $\mathrm{P}(13)$ holds.

- When $k = 12$, and $k + 1 = 13$:
  - Our IH assumes just $\mathrm{P}(12)$
  - In the IS, we say since $\mathrm{P}(9)$ holds (going back to $k - 3$), then $\mathrm{P}(13)$ holds.
  - But we don't know anything about $\mathrm{P}(9)$! It might not even be true!

- Lesson: If we go back s steps in the IS, we need s base cases.

Tower Visualization



[Proof by Strong Induction]

Prove that for all $n \geq 12$, I can make n ¢ worth of stamps.

1. Let $\mathrm{P}(n)$ be "I can make n ¢ worth of stamps with just 4¢ and 5¢ stamps." We prove $\mathrm{P}(n)$ for all integers $n \geq 12$ by strong induction.
2. Base Cases:
   12¢ can be made with three 4¢ stamps. Thus $\mathrm{P}(12)$ is true.
   13¢ can be made with two 4¢ stamps and one 5¢ stamps. Thus $\mathrm{P}(13)$ is true.
   14¢ can be made with one 4¢ stamp and two 5¢ stamps. Thus $\mathrm{P}(14)$ is true.
   15¢ can be made with three 5¢ stamps. Thus $\mathrm{P}(15)$ is true.
3. IH: Suppose $\mathrm{P}(12) \wedge \cdots \wedge \mathrm{P}(k)$ hold for an arbitrary integer $\boldsymbol{k \geq 15}$. I.e. we can make 12¢, 13¢, $\ldots, k$¢ worth of stamps with just 4¢ and 5¢ stamps.
4. IS: We aim to show $\mathrm{P}(k+1)$, i.e. that we can make $k+1$ cents in stamps. By the IH, we can make $k-3$ cents in stamps. Adding another 4¢ stamp gives exactly $k+1$ cents.
   [Note: Now $k+1 \geq 16$, so $k-3 \geq 12$. We're in the clear!]
5. Conclusion: Thus $\mathrm{P}(n)$ holds for all integers $n \geq 12$ by strong induction.

Strong Induction Lesson

Be careful about base cases!!

Strong Induction Template

Strong Induction Template

1. Define $\mathrm{P}(n)$. State that your proof is by strong induction on n.
2. Base Case: Show your base cases $\mathrm{P}(b_{\mathrm{min}}), \dots, \mathrm{P}(b_{\mathrm{max}})$ are true.
3. Inductive Hypothesis: Suppose $\mathrm{P}(b_{\mathrm{min}}) \wedge \cdots \wedge \mathrm{P}(k)$ hold for an arbitrary integer $k \geq b_{\mathrm{max}}$.
4. Inductive Step: Prove $\mathrm{P}(k+1)$ using the IH.
5. Conclusion: Conclude by saying $\mathrm{P}(n)$ holds for all integers $n \geq b_{\mathrm{min}}$ by strong induction.

Practical Tip

- If you aren’t sure how many steps you’ll go back, leave space for the base cases.
- Do the IH / IS, and then fill in the base cases later.

Strong Induction Example

Fibonacci Sequence

Fibonacci Numbers

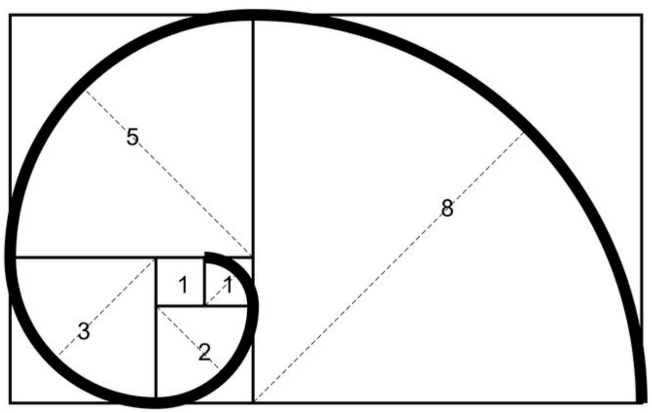
- The Fibonacci Numbers are defined as follows:

  $f_0 = 0$

  $f_1 = 1$

  $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

- I.e. $0, 1, 1, 2, 3, 5, 8, \ldots$



Fibonacci Numbers Claim

- We claim that $f_n < 2^n$ for all $n \geq 0$.
- $f_0 = 0$ $\quad\quad$ $2^0 = 1$
- $f_1 = 1$ $\quad\quad$ $2^1 = 2$
- $f_2 = 1$ $\quad\quad$ $2^2 = 4$
- $f_3 = 2$ $\quad\quad$ $2^3 = 8$
- $f_4 = 3$ $\quad\quad$ $2^4 = 16$

- We prove by strong induction!

Prove that for all $n \in \mathbb{N}$, $f_n < 2^n$.

Definition:
$f_0 = 0, f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

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Prove that for all $n \in \mathbb{N}$, $f_n < 2^n$.

1. Let $\mathrm{P}(n)$ be "$f_n < 2^n$" We prove $\mathrm{P}(n)$ for all $n \in \mathbb{N}$ by strong induction.
2. Base Cases:

Prove that for all $n \in \mathbb{N}$, $f_n < 2^n$.

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$f_0 = 0, f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

1. Let $\mathrm{P}(n)$ be "$f_n < 2^n$" We prove $\mathrm{P}(n)$ for all $n \in \mathbb{N}$ by strong induction.
2. Base Cases:
   $f_0 = 0$ and $2^0 = 1$. Since $0 < 1$, $\mathrm{P}(0)$ holds.
   $f_1 = 1$ and $2^1 = 2$. Since $1 < 2$, $\mathrm{P}(1)$ holds.
3. IH:

Prove that for all $n \in \mathbb{N}$, $f_n < 2^n$.

Definition:
$f_0 = 0, f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

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3. IH: Suppose $\mathrm{P}(0) \wedge \cdots \wedge \mathrm{P}(k)$ hold for an arbitrary integer $\boldsymbol{k \geq 1}$.
4. IS:

Prove that for all $n \in \mathbb{N}$, $f_n < 2^n$.

Definition:
$f_0 = 0, f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

1. Let $\mathrm{P}(n)$ be "$f_n < 2^n$" We prove $\mathrm{P}(n)$ for all $n \in \mathbb{N}$ by strong induction.
2. Base Cases:
   $f_0 = 0$ and $2^0 = 1$. Since $0 < 1$, $\mathrm{P}(0)$ holds.
   $f_1 = 1$ and $2^1 = 2$. Since $1 < 2$, $\mathrm{P}(1)$ holds.
3. IH: Suppose $\mathrm{P}(0) \wedge \cdots \wedge \mathrm{P}(k)$ hold for an arbitrary integer $\boldsymbol{k \geq 1}$.
4. IS: We aim to show $\mathrm{P}(k+1)$, i.e. that $f_{k+1} < 2^{k+1}$.

Prove that for all $n \in \mathbb{N}$, $f_n < 2^n$.

Definition:
$f_0 = 0, f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

1. Let $\mathrm{P}(n)$ be "$f_n < 2^n$" We prove $\mathrm{P}(n)$ for all $n \in \mathbb{N}$ by strong induction.
2. Base Cases:
   $f_0 = 0$ and $2^0 = 1$. Since $0 < 1$, $\mathrm{P}(0)$ holds.
   $f_1 = 1$ and $2^1 = 2$. Since $1 < 2$, $\mathrm{P}(1)$ holds.
3. IH: Suppose $\mathrm{P}(0) \wedge \cdots \wedge \mathrm{P}(k)$ hold for an arbitrary integer $\boldsymbol{k \geq 1}$.
4. IS: We aim to show $\mathrm{P}(k+1)$, i.e. that $f_{k+1} < 2^{k+1}$. Observe:

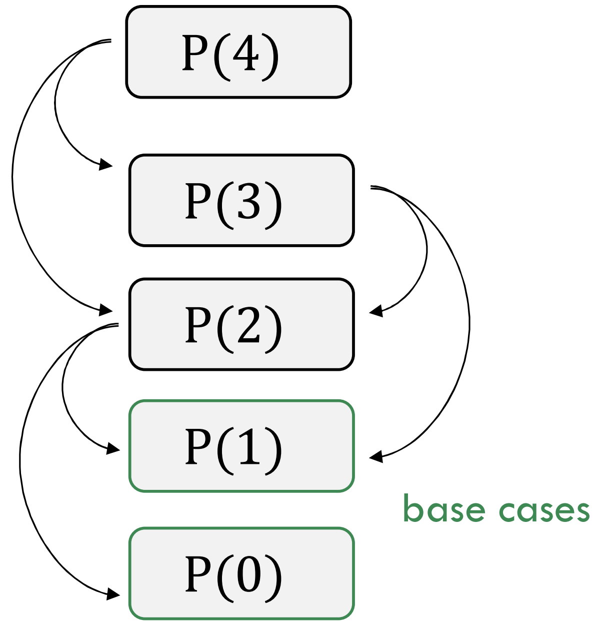
$$
\begin{aligned}
f_{k+1} &= f_k + f_{k-1} && \text{Since } k+1 \geq 2 \\
&\leq 2^k + f_{k-1} && \text{By IH, since } \mathrm{P}(k) \text{ is assumed} \\
&\leq 2^k + 2^{k-1} && \text{By IH, since } \mathrm{P}(k-1) \text{ is assumed} \\
&\leq 2^k + 2^k && \text{Since } 2^{k-1} = \tfrac{1}{2} \cdot 2^k \leq 2^k \\
&= 2^{k+1}
\end{aligned}
$$

Prove that for all $n \in \mathbb{N}$, $f_n < 2^n$.

Definition:
$f_0 = 0, f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

1. Let $\mathrm{P}(n)$ be "$f_n < 2^n$" We prove $\mathrm{P}(n)$ for all $n \in \mathbb{N}$ by strong induction.
2. Base Cases:
   $f_0 = 0$ and $2^0 = 1$. Since $0 < 1$, $\mathrm{P}(0)$ holds.
   $f_1 = 1$ and $2^1 = 2$. Since $1 < 2$, $\mathrm{P}(1)$ holds.
3. IH: Suppose $\mathrm{P}(0) \wedge \cdots \wedge \mathrm{P}(k)$ hold for an arbitrary integer $\boldsymbol{k \geq 1}$.
4. IS: We aim to show $\mathrm{P}(k+1)$, i.e. that $f_{k+1} < 2^{k+1}$. Observe:
   $f_{k+1} = f_k + f_{k-1}$ — Since $k+1 \geq 2$
   $\leq 2^k + f_{k-1}$ — By IH, since $\mathrm{P}(k)$ is assumed
   $\leq 2^k + 2^{k-1}$ — By IH, since $\mathrm{P}(k-1)$ is assumed
   $\leq 2^k + 2^k$ — Since $2^{k-1} = \frac{1}{2} \cdot 2^k \leq 2^k$
   $= 2^{k+1}$
5. Conclusion: Thus $\mathrm{P}(n)$ holds for all $n \in \mathbb{N}$ by strong induction.

Fibonacci Tower



How many base cases?

- Always at least one base case.
- If you're analyze a recursive function, at least one for each base case of the function.
- If you go back s steps in the proof, at least s base cases.

Definition:
$f_0 = 0, f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

Prove that for all $n \in \mathbb{N}$, $f_n < 2^n$.

1. Let $\mathrm{P}(n)$ be
2. Base Cases:
3. IH:
4. IS:
5. Conclusion: