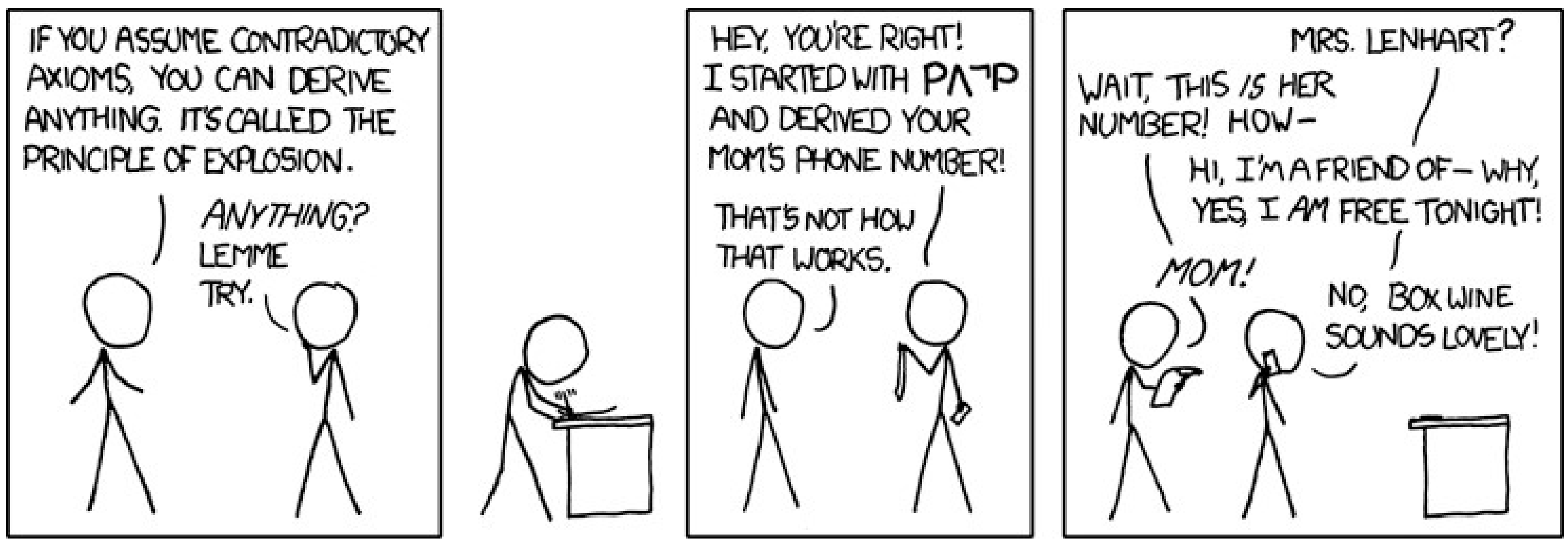


Number Theory

CSE 311
Lecture 7

Proof Style for Number Theory

- We use predicate logic to make the proof claim very precise.

However, please write the actual proofs in English, not logic!

- E.g. for all integers x, if x is odd then $x + 1$ is even.
- Good: Let x be arbitrary. Suppose x is odd. Then $x = 2k + 1$ for some integer k ...
- Bad: Let x be arbitrary. Suppose $\mathrm{Odd}(x)$. Then $\exists k\ (x = 2k + 1)$...

Number Theory: Motivation

Number Theory

- Branch of mathematics that deals with the properties and relationships of numbers
  - E.g. can we efficiently test if an integer is prime?
  - E.g. can we efficiently factor an integer?

- Many significant applications in computing
  - Cryptography & Security
  - Hashing

- Playground for practicing proof-writing

Modular Arithmetic

- Arithmetic over a finite domain
- In computing, almost all computations are over a finite domain

Modular Arithmetic

```
public class Test {
    final static int SEC_IN_YEAR = 365*24*60*60;
    public static void main(String args[]) {
        System.out.println( “I will be alive for at least ” +  SEC_IN_YEAR * 100 + “ seconds.” );
    }
}
```

Divisibility

Divisibility

- Definition:

For integers a, b, we say $a \mid b$ ("a divides b") iff there exists some integer k such that $b = ka$.

- Informally: "a fits into b" or "a is a factor of b"

- Examples: $5 \mid 15$ $\quad -3 \mid 9$ $\quad 5 \nmid 21$

Divisibility

Definition
$a \mid b := \exists k \in \mathbb{Z}\ (b = ka)$

- Which of these is true?

$5 \mid 1$ $\qquad$ $25 \mid 5$ $\qquad$ $7 \mid 0$ $\qquad$ $-2 \mid 4$

$1 \mid 5$ $\qquad$ $5 \mid 25$ $\qquad$ $0 \mid 7$ $\qquad$ $4 \mid -2$

Division Theorem

- Division Theorem:

For any integer a and positive integer d, there exist unique integers q, r with $0 \leq r < d$ such that $a = qd + r$.

Division Theorem

- Division Theorem:

For any integer a and positive integer d, there exist unique integers q, r with $0 \leq r < d$ such that $a = qd + r$.

- q is referred to as the quotient
- r is referred to as the remainder

Division Theorem

- Division Theorem:

For any integer a and positive integer d, there exist unique integers q, r with $0 \leq r < d$ such that $a = qd + r$.

- In Java, q is the result of the operation a/d
- In Java, r is the result of the operation $a \,\%\, d$

<u>Warning</u>
When dealing with negative numbers, Java's % may behave differently!

The mod (%) operator

Division Theorem
$a = qd + r$ with $0 \le r < d$

- The % operator is often referred to as "mod"
- $a \,\%\, d$ returns the remainder r when you divide a by d

- $22 \,\%\, 5 = 2$
- $22 = 4 \cdot 5 + \mathbf{2}$

$25 \,\%\, 5 = 0$

$25 = 5 \cdot 5 + \mathbf{0}$

- $0 \,\%\, 5 = 0$
- $0 = 0 \cdot 5 + \mathbf{0}$

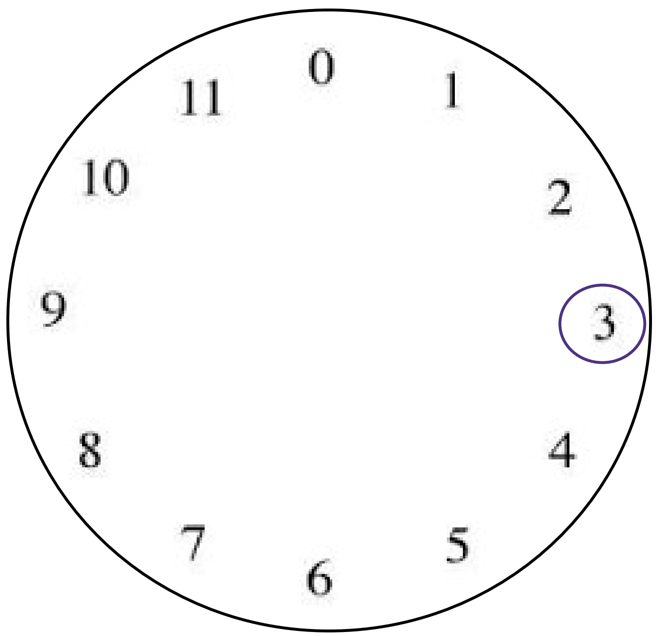
$-1 \,\%\, 4 = 3$

$-1 = -1 \cdot 4 + \mathbf{3}$

Modular Arithmetic

Modular Arithmetic: Like a Clock

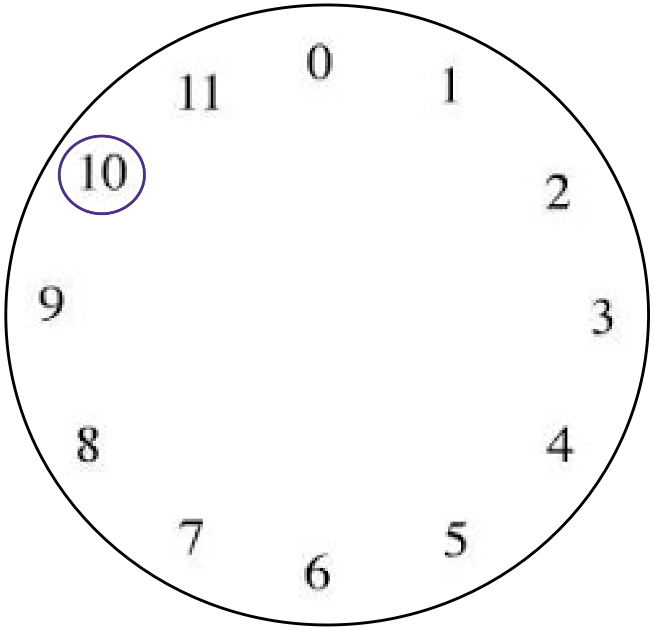
- Imagine you can only represent numbers $0, \ldots, 11$. We call this "arithmetic mod 12".
- What's $8 + 7$? 3



Observation
The solution is $a \ \% \ 12$.

Modular Arithmetic: Like a Clock

- Imagine you can only represent numbers $0, \dots, 11$. We call this "arithmetic mod 12".
- What's $3 - 5$? 10

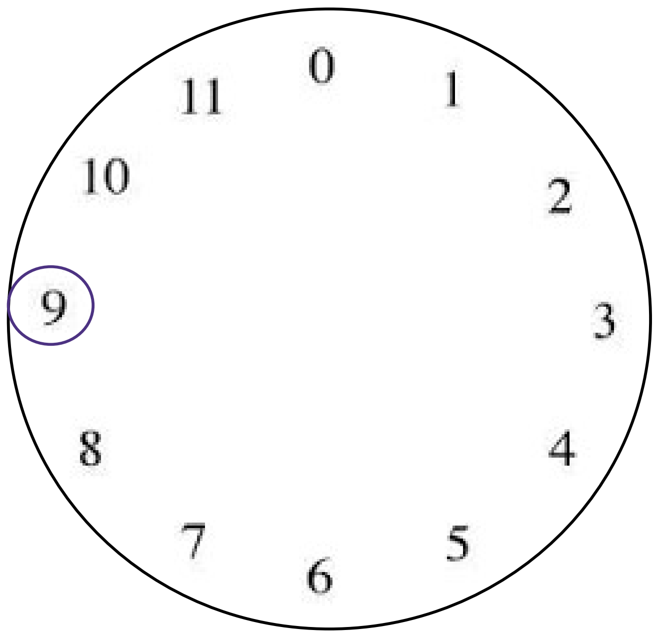


<u>Observation</u>
The solution is a % 12.

Modular Arithmetic: Like a Clock

- Imagine you can only represent numbers $0, \ldots, 11$. We call this "arithmetic mod 12".

What's $3 \cdot 7$? 9

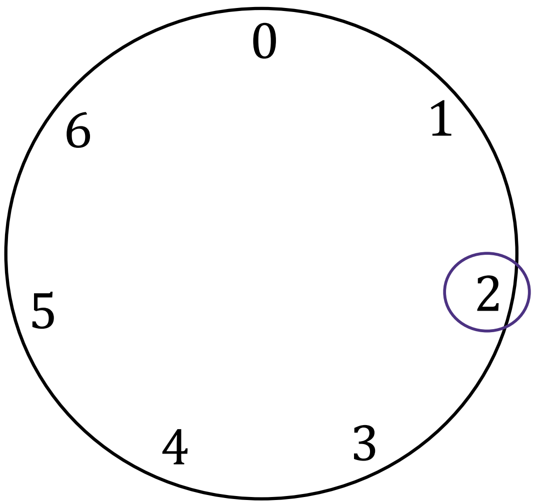


Observation
The solution is $a \ \% \ 12$.

Modular Arithmetic: Generalizing

- We can extend modular arithmetic to clocks of any positive integer size.

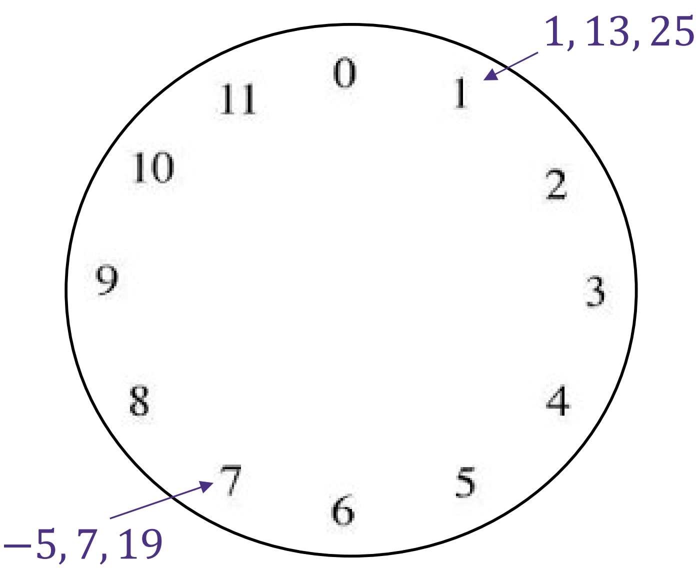
E.g. $3 + 6$ in arithmetic mod 7 is 2



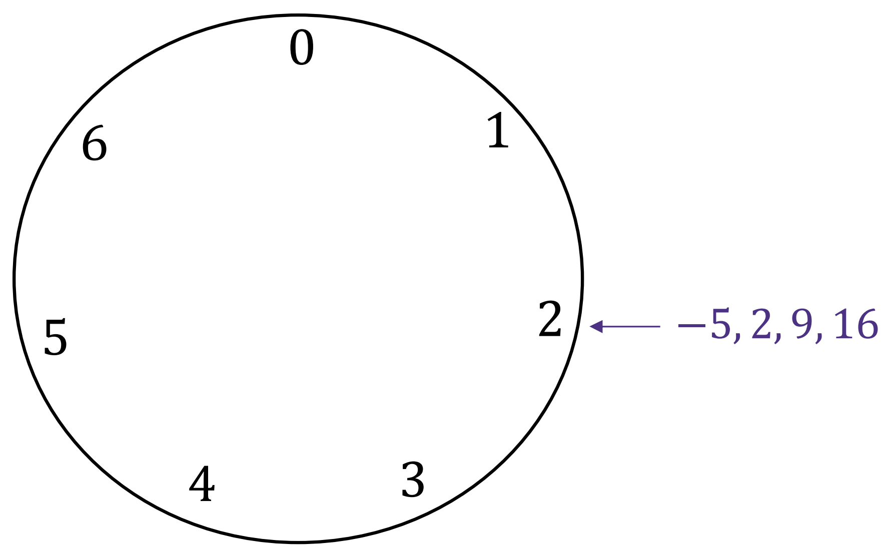
“Sameness”

- In modular arithmetic, many numbers have a notion of “sameness”.

Arithmetic mod 12:



Arithmetic mod 7:



“Sameness”

- In modular arithmetic, many numbers have a notion of “sameness”.

- To say “the same”, we don’t to use the $=$ symbol.

  E.g. $13 = 1$ is wrong…

- To say same in arithmetic mod m, we use the symbol $\equiv_m$
  - Pronounced “congruent mod m”
  - $13 \equiv_{12} 1$ $\quad$ $13 \equiv_{12} 25$ $\quad$ $2 \equiv_{12} 14$
  - $3 \equiv_7 10$ $\quad$ $0 \equiv_7 7$

Congruence

- We need a formal definition of $a \equiv_m b$.

We can't just say "a and b are on the same place in the m clock ☺"

- Definition:

For integers a, b and positive integer m, we say $a \equiv_m b$ iff $m \mid (a - b)$.

- Note: $a \equiv_m b$ is equivalent to $a \,\%\, m = b \,\%\, m$.

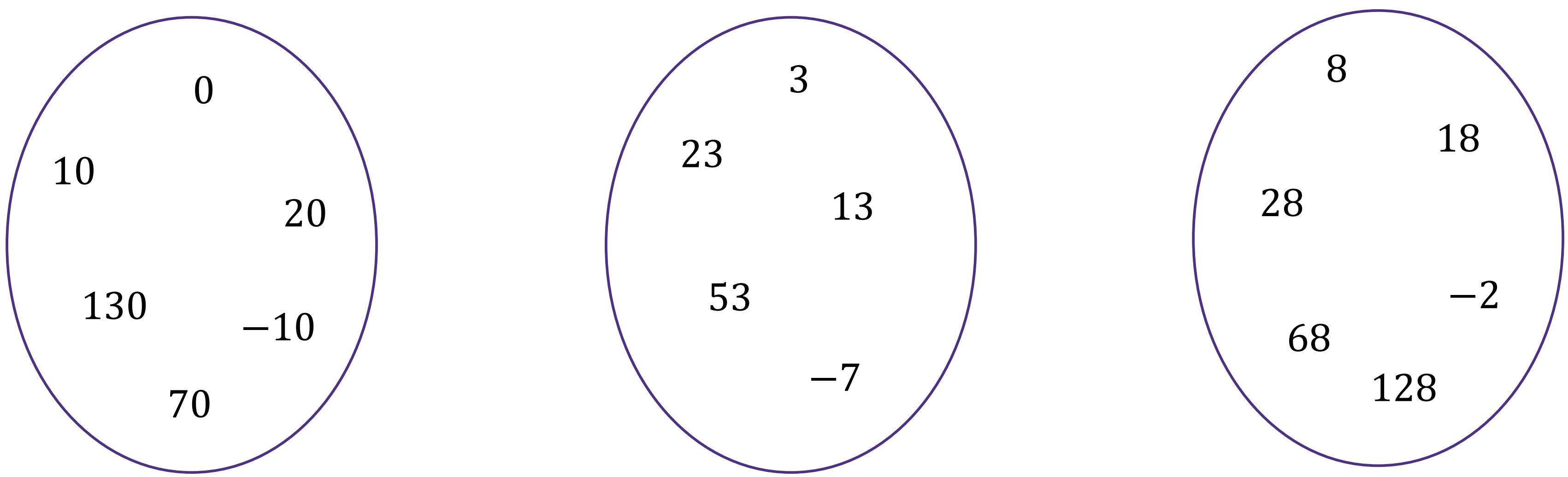
We will actually prove that the two notions are the same. But, the formal definition is much easier to use in proofs.

Intuition

Definition: $a \equiv_m b$ is defined as $m \mid (a - b)$

Intuition: Equivalently, $a \equiv_m b$ means $a \,\%\, m = b \,\%\, m$

- Here we have some groups of numbers that are congruent mod 10.



Congruent to 0 Congruent to 3 Congruent to 8

Properties of Congruence

Recall: Familiar Properties of $=$ in algebra

- If $a = b$, then $b = a$.
- If $a = b$ and $c = d$, then $a + c = b + d$.
- If $a = b$ and $c = d$, then $ac = bd$.
- If $a = b$ and $b = c$, then $a = c$.

- These are the facts that allow us to use algebra to solve problems.

We will prove analogous facts for modular arithmetic.

Claim 1

Definitions:

$a \mid b$ iff $\exists k \in \mathbb{Z}\ (b = ka)$

$a \equiv_m b$ iff $m \mid (a - b)$

Claim 1: For integers a, b and positive integer m, if $a \equiv_m b$ then $b \equiv_m a$.

Proof

Let a, b be arbitrary integers and let m be an arbitrary positive integer. Suppose that $a \equiv_m b$. Then by definition of congruence, $m \mid (a - b)$. Then by definition of divides, there exists some integer k such that $a - b = mk$. Then multiplying both sides by -1, we have $b - a = -mk = m(-k)$. Since k is an integer, $-k$ is an integer. So by definition of divides, $m \mid (b - a)$. Then by definition of congruence, $b \equiv_m a$. Since a, b, m were arbitrary, the claim holds.

Note on Claim 1

- You'll see $a \equiv_m b$ defined as $m \mid (a - b)$ or $m \mid (b - a)$ depending on where you look.
- Claim 1 proves these definitions are equivalent. From now on, you can use either definition in your proofs.
- In general, once we have proved claims in class, you can use those claims in your homework without proof.

Claim 2

Definitions:

$a \mid b$ iff $\exists k \in \mathbb{Z}\ (b = ka)$

$a \equiv_m b$ iff $m \mid (a - b)$

Claim 2: For integers a, b, c, d and positive integer m, if $a \equiv_m b$ and $c \equiv_m d$ then $a + c \equiv_m b + d$.

Intuition

$3 \equiv_{10} 13$ and $14 \equiv_{10} 24 \qquad \Rightarrow \qquad 17 \equiv_{10} 37$

Claim 2

Definitions:

$a \mid b$ iff $\exists k \in \mathbb{Z}\ (b = ka)$

$a \equiv_m b$ iff $m \mid (a - b)$

Claim 2: For integers a, b, c, d and positive integer m, if $a \equiv_m b$ and $c \equiv_m d$ then $a + c \equiv_m b + d$.

Proof

Let a, b, c, d and $m > 0$ be arbitrary integers. Suppose that $a \equiv_m b$ and $c \equiv_m d$. Then by definition of congruence, $m \mid (a - b)$ and $m \mid (c - d)$. Then by definition of divides, there exists some integers k, j such that $a - b = mk$ and $c - d = mj$. Then adding both expressions, we have:

$$a - b + c - d = mk + mj$$

$$(a + c) - (b + d) = m(k + j)$$

So by definition of divides, $m \mid (a + c) - (b + d)$. Then by definition of congruence, $a + c \equiv_m b + d$. Since a, b, c, d, m were arbitrary, the claim holds.

Claim 3

Definitions:

$a \mid b$ iff $\exists k \in \mathbb{Z}\ (b = ka)$

$a \equiv_m b$ iff $m \mid (a - b)$

Claim 3: For integers a, b, c, d and positive integer m, if $a \equiv_m b$ and $c \equiv_m d$ then $ac \equiv_m bd$.

Intuition

$2 \equiv_{10} 12$ and $3 \equiv_{10} 13 \Rightarrow \quad 6 \equiv_{10} 156$

Claim 3

Definitions:

$a \mid b$ iff $\exists k \in \mathbb{Z}\ (b = ka)$

$a \equiv_m b$ iff $m \mid (a - b)$

Claim 3: For integers a, b, c, d and positive integer m, if $a \equiv_m b$ and $c \equiv_m d$ then

$ac \equiv_m bd$.

Proof (Attempt 1)

Let a, b, c, d and $m > 0$ be arbitrary integers. Suppose that $a \equiv_m b$ and $c \equiv_m d$. Then by definition of congruence, $m \mid (a - b)$ and $m \mid (c - d)$. Then by definition of divides, there exists some integers k, j such that $a - b = mk$ and $c - d = mj$. Then multiplying both expressions, we have:

$$(a - b)(c - d) = mk \cdot mj$$

$$ac - bc - ad + bd = m^2 kj$$

$$??$$

DEAD END

Goal: $ac - bd = mx$

Claim 3

Definitions:

$a \mid b$ iff $\exists k \in \mathbb{Z}\ (b = ka)$

$a \equiv_m b$ iff $m \mid (a - b)$

Claim 3: For integers a, b, c, d and positive integer m, if $a \equiv_m b$ and $c \equiv_m d$ then $ac \equiv_m bd$.

Proof (Attempt 2):

Let a, b, c, d and $m > 0$ be arbitrary integers. Suppose that $a \equiv_m b$ and $c \equiv_m d$. Then by definition of congruence, $m \mid (a - b)$ and $m \mid (c - d)$. Then by definition of divides, there exists some integers k, j such that $a - b = mk$ and $c - d = mj$. Rearranging, we have $a = mk + b$ and $c = mj + d$. Multiplying both expressions, we have:

$$ac = (mk + b)(mj + d)$$

$$ac = m^2kj + mbj + mdk + bd$$

$$ac - bd = m^2kj + mbj + mdk$$

$$ac - bd = m(mkj + bj + dk)$$

Since m, k, j, b, d are integers, $mkj + bj + dk$ is an integer. Thus by definition of divides, $m \mid ac - bd$. Then by definition of congruence, $ac \equiv_m bd$. Since a, b, c, d, m were arbitrary, the claim holds.

Claim 4

Definitions:

$a \mid b$ iff $\exists k \in \mathbb{Z}\ (b = ka)$

$a \equiv_m b$ iff $m \mid (a - b)$

- : For integers a, b, c and positive integer m, if $a \equiv_m b$ and $b \equiv_m c$ then $a \equiv_m c$.

Proof

Let a, b, c and $m > 0$ be arbitrary integers. Suppose that $a \equiv_m b$ and $b \equiv_m c$. Then by definition of congruence, $m \mid (a - b)$ and $m \mid (b - c)$. Then by definition of divides, there exists some integers k, j such that $a - b = mk$ and $b - c = mj$. Adding the expressions, we have:

$$(a - b) + (b - c) = mk + mj$$
$$a - c = m(k + j)$$

Since k, j are integers, $k + j$ is an integer. Thus by definition of divides, $m \mid a - c$. Then by definition of congruence, $a \equiv_m c$. Since a, b, c, m were arbitrary, the claim holds.

Claim 5:

Claim 5: For integers a, b and $m > 0$, $a \equiv_m b$ if and only if $a \,\%\, m = b \,\%\, m$.

For integers a, b and $m > 0$, $a \equiv_m b$ if and only if $a \,\%\, m = b \,\%\, m$.

$\Rightarrow$ Let $a, b, m > 0$ be arbitrary integers, and suppose $a \equiv_m b$. Then $m \mid (a - b)$. So there exists some integer k such that $a - b = km$. So $a = km + b$.

By the Division Theorem, $a = qm + (a \,\%\, m)$ for some integer q, where $0 \leq a \,\%\, m < m$. Thus:

$$km + b = qm + (a \,\%\, m)$$

$$b = qm - km + (a \,\%\, m)$$

$$b = (q - k)m + (a \,\%\, m)$$

By the Division Theorem again, we have that $b \,\%\, m = a \,\%\, m$.

Since a, b, m were arbitrary, the claim holds.

> For integers a, b and $m > 0$, $a \equiv_m b$ if and only if $a \,\%\, m = b \,\%\, m$.

$\Leftarrow$ Let $a, b, m > 0$ be arbitrary integers, and suppose $a \,\%\, m = b \,\%\, m$. By the Division Theorem, $a = mq + (a \,\%\, m)$ for some integer q, and $b = ms + (b \,\%\, m)$ for some integer s. Thus:

$$a - b = \big(mq + (a \,\%\, m)\big) - \big(ms + (b \,\%\, m)\big)$$

$$a - b = mq - ms + (a \,\%\, m) - (b \,\%\, m)$$

$$a - b = m(q - s)$$

Since q, s are integers, $q - s$ is an integer. So $m \mid (a - b)$. So $a \equiv_m b$.
Since a, b, m were arbitrary, the claim holds.

Summary: Properties of Mod

- Let a, b, c, d and $m > 0$ be integers.

- If $a \equiv_m b$, then $b \equiv_m a$.
- If $a \equiv_m b$ and $c \equiv_m d$, then $a + c \equiv_m b + d$.
- If $a \equiv_m b$ and $c \equiv_m d$, then $ac \equiv_m bd$.
- If $a \equiv_m b$ and $b \equiv_m c$, then $a \equiv_m c$.
- $a \equiv_m b$ if and only if $a \,\%\, m = b \,\%\, m$.