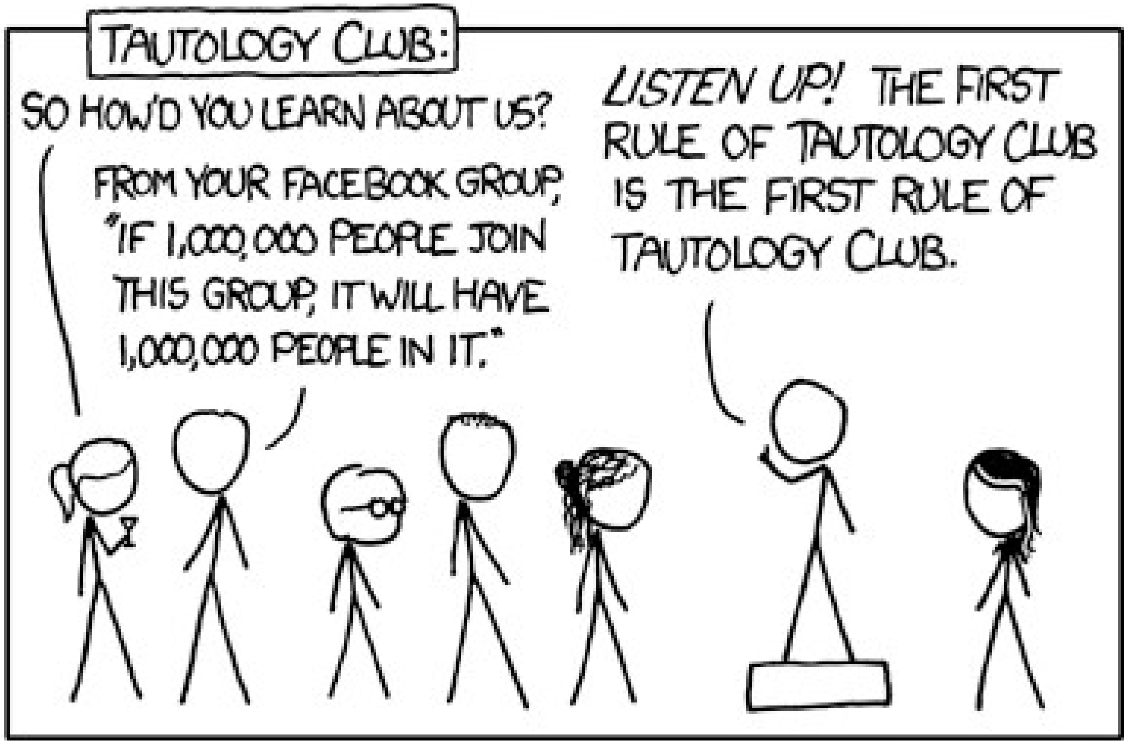


Predicate Logic, Direct Proofs

CSE 311
Lecture 4

Announcements

- Homework 1 is due Today
  - Make sure you select each questions part!
- Homework 2 is out some time later today

Predicate Logic

Motivation

- Often we will work with statements of the form:
  - If $x > 10$, then $x^2 > 100$.
  - If x is even, x^2 is even.

- Can you translate these to propositional logic?
- No. We need a <u>function</u> that is true or false depending on the value of x.

Motivation

- Propositional Logic
- Lets us break down complex true or false statements into atomic parts joined by connectives.

- Predicate Logic
- Lets us analyze complex true or false statements that are functions of some underlying objects.

Predicate Logic Summary

- 3 Parts

1. Predicate – Function that outputs true or false.

2. Domain of Discourse – Possible inputs to a predicate statement.

3. Quantifiers – A statement about when a predicate is true: $\forall$ or $\exists$

Translation to English

- Translations often sound more natural if we:
  1. Notice domain restriction patterns.
  2. Avoid using variables when we can.
  3. Drop the “for all” or “there exists” when we can.

- For example:
- $\forall x(\mathrm{Cat}(x) \rightarrow \mathrm{Blue}(x))$
- ❌ For all animals x, if x is a cat then x is blue.
- ✅ All cats are blue.

Translation to English

Domain of Discourse
Food

Predicate Definitions
$\text{Fruit}(x) \coloneqq x$ is a fruit
$\text{Tasty}(x) \coloneqq x$ is tasty
$\text{Ripe}(x) \coloneqq x$ is ripe

- Translate these sentences using a natural-sounding translation.
- $\exists x\big(\text{Fruit}(x) \wedge \text{Tasty}(x)\big)$
- There is a tasty fruit. OR Some fruits are tasty.

- $\forall x\Big(\big(\text{Fruit}(x) \wedge \neg\text{Ripe}(x)\big) \rightarrow \neg\text{Tasty}(x)\Big)$
- All fruits that aren't ripe aren't tasty.

Quantifier Scope

- $\exists x\, \big(\mathrm{P}(x) \wedge \mathrm{Q}(x)\big)$ vs. $\exists x\, \mathrm{P}(x) \wedge \exists x\, \mathrm{Q}(x)$

Could be different x's

<u>For example</u>
Domain of Discourse: Integers
$\mathrm{P}(x) :=$ x is odd
$\mathrm{Q}(x) :=$ x is even

Nested Quantifiers

Example 1

<u>Domain of Discourse</u>
Mammals

<u>Predicate Definitions</u>
$\text{Walks}(x, y) \coloneqq x$ walks y
$\text{Friends}(x, y) \coloneqq x$ and y are friends
$\text{Human}(x) \coloneqq x$ is a human
$\text{Dog}(x) \coloneqq x$ is a dog

All humans are friends with the dogs that they walk.

$$\forall x \forall y \left(\left(\text{Human}(x) \land \text{Dog}(y) \land \text{Walks}(x, y) \right) \rightarrow \text{Friends}(x, y) \right)$$

Example 2

Domain of Discourse
Mammals

Predicate Definitions
$\text{Walks}(x, y) \coloneqq x$ walks y
$\text{Friends}(x, y) \coloneqq x$ and y are friends
$\text{Human}(x) \coloneqq x$ is a human
$\text{Dog}(x) \coloneqq x$ is a dog

Every human walks a dog.

$$\forall \boldsymbol{x} \exists \boldsymbol{y} \left(\textbf{Human}(\boldsymbol{x}) \rightarrow \left(\textbf{Dog}(\boldsymbol{y}) \land \textbf{Walks}(\boldsymbol{x}, \boldsymbol{y}) \right) \right)$$

Example 3

Domain of Discourse
Mammals

Predicate Definitions
$\mathrm{Walks}(x, y) \coloneqq x$ walks y
$\mathrm{Friends}(x, y) \coloneqq x$ and y are friends
$\mathrm{Human}(x) \coloneqq x$ is a human
$\mathrm{Dog}(x) \coloneqq x$ is a dog

Every human walks exactly one dog.

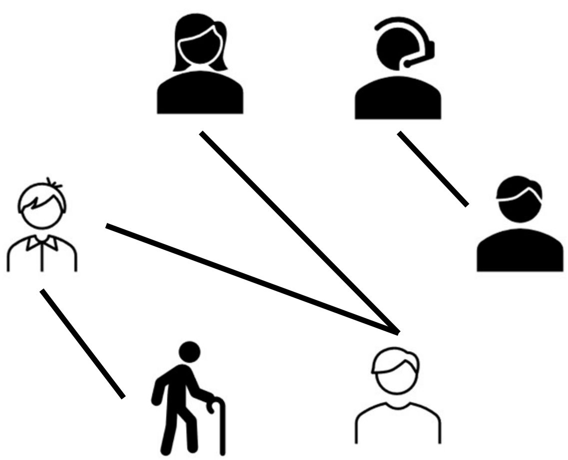
$$\forall x[\mathrm{Human}(x) \to \exists y(\mathrm{Dog}(y) \land \mathrm{Walks}(x, y) \land \forall z(\mathrm{Dog}(z) \land (z \neq y) \to \neg \mathrm{Walks}(x, z)))]$$

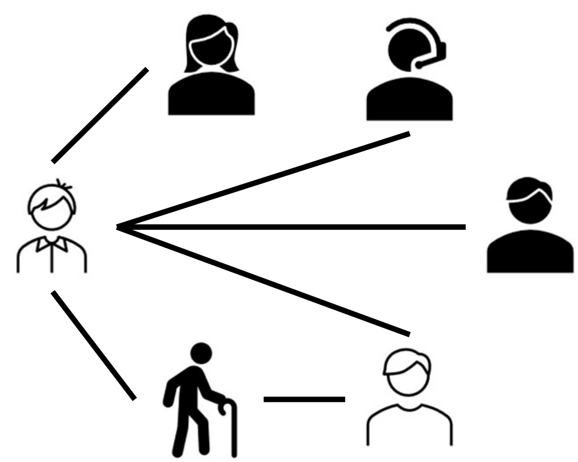
Quantifier Order

Quantifier Order

Translate to logic. The domain of discourse is people. The predicate $\texttt{Friends}(x, y)$ is defined as x and y are friends.

- Everyone is friends with someone.
- Someone is friends with everyone.





$\forall x \exists y\ \texttt{Friends}(x, y)$

$\exists y \forall x\ \texttt{Friends}(x, y)$

Quantifier Order

$\forall x \exists y\, P(x, y)$ means for every x, there exists a y such that $P(x, y)$ is true.

$\exists y \forall x\, P(x, y)$ means there exists some y such that for all x, $P(x, y)$ is true.

Quantifier Order

$\forall x \exists y \text{ Likes}(x, y)$

Everyone has some person they like.

$\forall y \exists x \text{ Likes}(x, y)$

Everyone has some person that likes them.

$\exists x \forall y \text{ Likes}(x, y)$

There is a person that likes everyone.

$\exists y \forall x \text{ Likes}(x, y)$

There is a person that everyone likes.

Quantifier Order

- Let our domain of discourse be $\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}\}$
- And our proposition $\mathrm{P}(x, y)$ be given by the table.
- What should we look for in the table?
- $\exists x \forall y \; \mathrm{P}(x, y)$
- $\forall x \exists y \; \mathrm{P}(x, y)$

y

x

$P(x, y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Quantifier Order

- Let our domain of discourse be $\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}\}$
- And our proposition $\mathrm{P}(x, y)$ be given by the table.
- What should we look for in the table?
- $\exists x \forall y\ \mathrm{P}(x, y)$

  A row, where every entry is `T`
- $\forall x \exists y\ \mathrm{P}(x, y)$

  In every row there must be a `T`

y (columns), x (rows)

$P(x,y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Negating Quantifiers

DeMorgan's Law for Quantifiers

- Consider the following sentences:
- There does not exist a green penguin.
- Every penguin is a color other than green.

- Are they logically equivalent?

DeMorgan’s Law for Quantifiers

- Consider the following sentences:
  - Not every person can dance.
  - There is a person that cannot dance.

- Are they logically equivalent?

DeMorgan's Law for Quantifiers

$$\boxed{\begin{array}{c}\neg\forall x\ P(x) \equiv \exists x\ \neg P(x) \\ \neg\exists x\ P(x) \equiv \forall x\ \neg P(x)\end{array}}$$

I.e. to negate an expression with a quantifier:

1. Switch the quantifier (∀ becomes ∃, and vice versa).
2. Negate the expression inside.

Example 1

- Translate to predicate logic & rewrite using DeMorgan's Law.

There is no integer which is prime and even.

$$\neg\exists x\big(\mathrm{Prime}(x) \wedge \mathrm{Even}(x)\big)$$

$$\equiv \forall x \neg\big(\mathrm{Prime}(x) \wedge \mathrm{Even}(x)\big)$$

$$\equiv \forall x\big(\neg\mathrm{Prime}(x) \vee \neg\mathrm{Even}(x)\big)$$

All integers are not prime or not even.

Example 2

- Translate to predicate logic & rewrite using DeMorgan's Law.

There is no integer greater than or equal to every other integer.

$$\neg\exists x\, \forall y\, (x \geq y)$$

$$\equiv \forall x\, \neg\, \forall y\, (x \geq y)$$

$$\equiv \forall x\, \exists y\, \neg(x \geq y)$$

$$\equiv \forall x\, \exists y\, (x < y)$$

For every integer, there is an integer greater than it.

Predicate Logic Equivalence

Motivation

- We saw with the last two examples that there may be different predicate logic expressions that have the same meaning
- We can prove logical equivalence of Predicate Logic statements like we did for Propositional Logic
- Same equivalence rules still apply, in addition to DeMorgan's Law for Quantifiers

Proving Predicate Logic Equivalence

- "No odd integer is equal to an even integer."
- Alice translated this as: $\neg\exists x\exists y\ \big(\mathrm{Odd}(x) \wedge \mathrm{Even}(y) \wedge (x = y)\big)$
-

Bob translated this as: $\forall x\forall y\big(\mathrm{Odd}(x) \wedge \mathrm{Even}(y) \rightarrow (x \neq y)\big)$

- Prove that these translations are logically equivalent.

Proving Predicate Logic Equivalence

$$\neg\exists x\exists y\left(\mathrm{Odd}(x)\land\mathrm{Even}(y)\land(x=y)\right)$$

$\equiv \forall x\neg\exists y\left(\mathrm{Odd}(x)\land\mathrm{Even}(y)\land(x=y)\right)$ DeMorgan's Law for Quantifiers

$\equiv \forall x\,\forall y\,\neg\left(\mathrm{Odd}(x)\land\mathrm{Even}(y)\land(x=y)\right)$ DeMorgan's Law for Quantifiers

$\equiv \forall x\,\forall y\,\left(\neg\left(\mathrm{Odd}(x)\land\mathrm{Even}(y)\right)\lor\neg(x=y)\right)$ DeMorgan's Law

$\equiv \forall x\forall y\left(\neg\left(\mathrm{Odd}(x)\land\mathrm{Even}(y)\right)\lor(x\neq y)\right)$ Definition of $\neq$

$\equiv \forall x\forall y\left(\left(\mathrm{Odd}(x)\land\mathrm{Even}(y)\right)\to(x\neq y)\right)$ Law of Implication

Theorems and Proofs

Theorems and Proofs

- Definition:
- A theorem is a statement that can be shown to be true.

- A proof is a valid argument that establishes a statement to be true.

Theorems and Proofs

- Examples of theorems include…

- Given a right triangle with side lengths a, b and hypotenuse c, $a^2 + b^2 = c^2$
- There are infinitely many prime numbers.
- There exists a problem that cannot be solved by a program.

Integer

- Definition:
- An integer is a real number with no fractional part.

e.g. $-17, 0, 1, 53$

Odd and Even

Definitions:

- An integer x is even iff there exists an integer k such that $x = 2k$.

$\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$

- An integer x is odd iff there exists an integer k such that $x = 2k + 1$.

$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Arbitrary Variables

Proof Strategy: Direct Proof

Direct Proof

- Direct proof is one strategy for proving statements of the form
- $\forall x[P(x) \rightarrow Q(x)]$

Our First Direct Proof

- Prove: "For all integers x, if x is even, then x^2 is even."

- What's the claim in logic?

- How would we prove this claim?
- We'll see how to prove it formally in a minute; for now, just try to convince each other this statement is true.

Our First Direct Proof

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

- Prove: "For all integers x, if x is even, then x^2 is even." $\forall x\big(\text{Even}(x) \to \text{Even}(x^2)\big)$

Arbitrary

- An "arbitrary" variable is one that
  - is part of the domain of discourse (or some sub-domain you pick).
  - You know **nothing** else about.
- EVERY element of the domain could be plugged into that arbitrary variable. And everything else you say in the proof will follow.
- An arbitrary variable is exactly what you need to convince us of a $\forall$.
- If you want to prove a for-all you must explicitly tell us the variable is arbitrary.
  - Your reader doesn't know what you're doing otherwise.

Our First Direct Proof

<u>Definitions</u>
$\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$

- Prove: "For all integers x, if x is even, then x^2 is even." $\forall x\big(\mathrm{Even}(x) \rightarrow \mathrm{Even}(x^2)\big)$
- Proof: Let x be an arbitrary integer. Suppose that x is even.

Now What?

<u>Definitions</u>
$\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$

- Well....what does it mean to be even?

- $x = 2k$ for some integer k.

- Where do we need to end up?
- $\texttt{Even}(x^2)$

Our First Direct Proof

<u>Definitions</u>
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

- Prove: "For all integers x, if x is even, then x^2 is even." $\forall x\big(\text{Even}(x) \to \text{Even}(x^2)\big)$
- Proof: Let x be an arbitrary integer. Suppose that x is even.
- So x^2 is even.

Our First Direct Proof

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

- Prove: "For all integers x, if x is even, then x^2 is even." $\forall x\big(\text{Even}(x) \rightarrow \text{Even}(x^2)\big)$
- Proof: Let x be an arbitrary integer. Suppose that x is even.
- By definition of even, $x = 2k$ for some integer k.
- So x^2 is even.

Our First Direct Proof

<u>Definitions</u>
$\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$

Prove: "For all integers x, if x is even, then x^2 is even." $\forall x\big(\mathrm{Even}(x) \to \mathrm{Even}(x^2)\big)$

Proof: Let x be an arbitrary integer. Suppose that x is even.

By definition of even, $x = 2k$ for some integer k.

Squaring both sides, we see that:

$$x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$$

Because k is an integer, $2k^2$ is also an integer.

So x^2 is two times an integer.

Which is exactly the definition of even, so x^2 is even.

Since x was an arbitrary integer, we conclude that for all integers x, if x is even then x^2 is also even.

Direct Proof Template

Prove: $\forall x\big(\mathrm{Even}(x) \to \mathrm{Even}(x^2)\big)$

Declare an arbitrary variable for each ∀.	Let x be an arbitrary integer.
Assume the left side of the implication.	Suppose that x is even.
Unroll the predicate definitions.	Then by definition of even, there exists some integer k such that $x = 2k$.
Manipulate towards the goal.	Squaring both sides, we see that: $x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$
Reroll definitions into the right side of the implication.	Because k is an integer, then $2k^2$ is also an integer. So x^2 is two times an integer. So by definition of even, x^2 is even.
Conclude that you have proved the claim.	Since x was an arbitrary integer, we can conclude that for all integers x, if x is even then x^2 is even.

Direct Proof Steps

These are the usual steps. We’ll see different outlines in the future!!

- Introduction
  - Declare an arbitrary variable for each $\forall$ quantifier
  - Assume the left side of the implication
- Core of the proof
  - Unroll the predicate definitions
  - Manipulate towards the goal (using creativity, algebra, etc.)
  - Reroll definitions into the right side of the implication
- Conclude that you have proved the claim

Another Direct Proof

- Prove: "The product of two odd integers is odd."

- What's the claim in logic? $\forall x \forall y \left(\left(\mathrm{Odd}(x) \land \mathrm{Odd}(y) \right) \rightarrow \mathrm{Odd}(xy) \right)$

- How would we prove this claim?

Direct Proof. In particular, we'll let x, y be arbitrary integers. We'll suppose x, y are odd. We'll show that $x \cdot y$ is odd.

Another Direct Proof

<u>Definitions</u>
$\text{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

- Prove: "The product of two odd integers is odd."

$$\forall x \forall y \Big(\big(\text{Odd}(x) \wedge \text{Odd}(y)\big) \rightarrow \text{Odd}(xy)\Big)$$

Let x and y be arbitrary integers. Suppose that x and y are odd. Then by definition of odd, there exists some integer k such that $x = 2k + 1$, and some integer j such that $y = 2j + 1$.

Then multiplying x and y, we can see that:

$$xy = (2k + 1) \cdot (2j + 1) = 4kj + 2j + 2k + 1 = 2(2kj + j + k) + 1$$

Since k, j are integers, $2kj + j + k$ is an integer. So by definition of odd, xy is odd. Since x, y were arbitrary, we have shown that the product of two odd integers is odd.

A note on Domain of Discourse

"The product of two odd integers is odd."

Domain: Integers

Translation:

$\forall x \forall y \left(\left(\mathrm{Odd}(x) \land \mathrm{Odd}(y) \right) \rightarrow \mathrm{Odd}(xy) \right)$

Proof Outline:

Let x and y be arbitrary integers.

Suppose x and y are odd.

Show xy is odd.

Domain: Odd Integers

Translation:

$\forall x \forall y \left(\mathrm{Odd}(xy) \right)$

Proof Outline:

Let x and y be arbitrary odd integers.

Show xy is odd.

A note on Translation to Logic

- We first translate the claim to predicate logic because:
  - The translation makes it precise what we are proving
  - The translation hints at the structure of the proof

    e.g. for each ∀, introduce an arbitrary variable
- In practice, computer scientists identify the proof claim and structure without predicate logic translation
- Eventually we'll stop asking you to translate to logic first

Square

- Definition:
- An integer x is square iff there exists an integer k such that $x = k^2$.

- $\text{Square}(x) \coloneqq \exists k\ (x = k^2)$

Yet Another Direct Proof

Definitions
$\text{Square}(x) \coloneqq \exists k\ (x = k^2)$

Prove: The product of two square integers is square.

What's the claim in logic?

$$\forall n \forall m \Big(\big(\text{Square}(n) \land \text{Square}(m)\big) \to \text{Square}(nm) \Big)$$

Prove this claim.

Yet Another Direct Proof

<u>Definitions</u>
$\text{Square}(x) \coloneqq \exists k\ (x = k^2)$

Prove: "The product of two square integers is square."

$$\forall n \forall m \left(\left(\text{Square}(n) \wedge \text{Square}(m) \right) \rightarrow \text{Square}(nm) \right)$$

Let n and m be arbitrary integers. Suppose that n and m are square. Then by definition of square, $n = k^2$ for some integer k, and $m = j^2$ for some integer j.

Then multiplying n and m, we can see:

$$nm = k^2 \cdot j^2 = (kj)^2$$

Since k and j are integers, kj is an integer. So by definition of square, nm is square. Since n and m were arbitrary, we have shown that the product of two square integers is square.