Predicate Logic,
Direct Proofs

CSE 311
Lecture 4
Announcements

• Homework 1 is due Today
  - Make sure you select each questions part!
• Homework 2 is out some time later today
Predicate Logic
Motivation

- Often we will work with statements of the form:
  - If $x > 10$, then $x^2 > 100$.
  - If $x$ is even, $x^2$ is even.

- Can you translate these to propositional logic?
- No. We need a function that is true or false depending on the value of $x$. 
Motivation

- Propositional Logic
  - Lets us break down complex true or false statements into atomic parts joined by connectives.

- Predicate Logic
  - Lets us analyze complex true or false statements that are functions of some underlying objects.
Predicate Logic Summary

- 3 Parts
  1. Predicate – Function that outputs true or false.
  2. Domain of Discourse – Possible inputs to a predicate statement.
  3. Quantifiers – A statement about when a predicate is true: ∀ or ∃
Translation to English

- Translations often sound more natural if we:
  1. Notice domain restriction patterns.
  2. Avoid using variables when we can.
  3. Drop the “for all” or “there exists” when we can.

- For example:
  - $\forall x (\text{Cat}(x) \rightarrow \text{Blue}(x))$
  - $\times$ For all animals $x$, if $x$ is a cat then $x$ is blue.
  - $\checkmark$ All cats are blue.
- Translate these sentences using a natural-sounding translation.
- \( \exists x (\text{Fruit}(x) \land \text{Tasty}(x)) \)
- There is a tasty fruit. OR Some fruits are tasty.

- \( \forall x \left( (\text{Fruit}(x) \land \neg \text{Ripe}(x)) \rightarrow \neg \text{Tasty}(x) \right) \)
- All fruits that aren’t ripe aren’t tasty.
Quantifier Scope

• $\exists x \ (P(x) \land Q(x))$ vs. $\exists x \ P(x) \land \exists x \ Q(x)$

For example

Domain of Discourse: Integers
$P(x) := x$ is odd
$Q(x) := x$ is even

Could be different $x$’s
Nested Quantifiers
Example 1

All humans are friends with the dogs that they walk.

\[ \forall x \forall y \left( (\text{Human}(x) \land \text{Dog}(y) \land \text{Walks}(x, y)) \rightarrow \text{Friends}(x, y) \right) \]
Example 2

Every human walks a dog.

\[ \forall x \exists y \left( \text{Human}(x) \rightarrow (\text{Dog}(y) \land \text{Walks}(x, y)) \right) \]
Every human walks exactly one dog.

∀x[Human(x) → ∃y(Dog(y) ∧ Walks(x, y) ∧ ∀z(Dog(z) ∧ (z ≠ y) → ¬Walks(x, z)))]
Quantifier Order
Quantifier Order

Translate to logic. The domain of discourse is people. The predicate $\text{Friends}(x, y)$ is defined as $x$ and $y$ are friends.

Everyone is friends with someone.  

\[ \forall x \exists y \, \text{Friends}(x, y) \]

Someone is friends with everyone.

\[ \exists y \forall x \, \text{Friends}(x, y) \]
Quantifier Order

\( \forall x \exists y \ P(x, y) \) means for every \( x \), there exists a \( y \) such that \( P(x, y) \) is true.

\( \exists y \forall x \ P(x, y) \) means there exists some \( y \) such that for all \( x \), \( P(x, y) \) is true.
Quantifier Order

∀x∃y Likes(x, y)
Everyone has some person they like.

∃x∀y Likes(x, y)
There is a person that likes everyone.

∀y∃x Likes(x, y)
Everyone has some person that likes them.

∃y∀x Likes(x, y)
There is a person that everyone likes.
Quantifier Order

- Let our domain of discourse be \{A, B, C, D, E\}
- And our proposition \(P(x, y)\) be given by the table.
- What should we look for in the table?
- \(\exists x \forall y \ P(x, y)\)
- \(\forall x \exists y \ P(x, y)\)

<table>
<thead>
<tr>
<th>(P(x, y))</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>T</td>
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</tr>
</tbody>
</table>
Quantifier Order

• Let our domain of discourse be \{A, B, C, D, E\}
• And our proposition \(P(x, y)\) be given by the table.
• What should we look for in the table?
  • \(\exists x \forall y \ P(x, y)\)
    A row, where every entry is \(T\)
  • \(\forall x \exists y \ P(x, y)\)
    In every row there must be a \(T\)
DeMorgan’s Law for Quantifiers

- Consider the following sentences:
  - There does not exist a green penguin.
  - Every penguin is a color other than green.

- Are they logically equivalent?
DeMorgan’s Law for Quantifiers

- Consider the following sentences:
  - Not every person can dance.
  - There is a person that cannot dance.

- Are they logically equivalent?
DeMorgan’s Law for Quantifiers

\[ \neg \forall x \ P(x) \equiv \exists x \ \neg P(x) \]
\[ \neg \exists x \ P(x) \equiv \forall x \ \neg P(x) \]

I.e. to negate an expression with a quantifier:

1. Switch the quantifier (\(\forall\) becomes \(\exists\), and vice versa).
2. Negate the expression inside.
Example 1

- Translate to predicate logic & rewrite using DeMorgan’s Law.

There is no integer which is prime and even.

\[ \neg \exists x (\text{Prime}(x) \land \text{Even}(x)) \]

\[ \equiv \forall x \neg (\text{Prime}(x) \land \text{Even}(x)) \]

\[ \equiv \forall x (\neg \text{Prime}(x) \lor \neg \text{Even}(x)) \]

All integers are not prime or not even.
Example 2

- Translate to predicate logic & rewrite using DeMorgan’s Law.

There is no integer greater than or equal to every other integer.

$$\neg \exists x \forall y (x \geq y)$$

$$\equiv \forall x \neg \forall y (x \geq y)$$

$$\equiv \forall x \exists y \neg (x \geq y)$$

$$\equiv \forall x \exists y (x < y)$$

For every integer, there is an integer greater than it.
Predicate Logic Equivalence
Motivation

• We saw with the last two examples that there may be different predicate logic expressions that have the same meaning.
• We can prove logical equivalence of Predicate Logic statements like we did for Propositional Logic.
• Same equivalence rules still apply, in addition to DeMorgan’s Law for Quantifiers.
Proving Predicate Logic Equivalence

- “No odd integer is equal to an even integer.”
- Alice translated this as: \( \neg \exists x \exists y \left( \text{Odd}(x) \land \text{Even}(y) \land (x = y) \right) \)
- Bob translated this as: \( \forall x \forall y \left( \text{Odd}(x) \land \text{Even}(y) \rightarrow (x \neq y) \right) \)
- Prove that these translations are logically equivalent.
Proving Predicate Logic Equivalence

\[ \neg \exists x \exists y \ (\text{Odd}(x) \land \text{Even}(y) \land (x = y)) \]

\[ \equiv \forall x \neg \exists y \ (\text{Odd}(x) \land \text{Even}(y) \land (x = y)) \]

\[ \equiv \forall x \forall y \ (\neg(\text{Odd}(x) \land \text{Even}(y)) \lor \neg(x = y)) \]

\[ \equiv \forall x \forall y \ (\neg(\text{Odd}(x) \land \text{Even}(y)) \lor (x \neq y)) \]

\[ \equiv \forall x \forall y \ (\text{Odd}(x) \land \text{Even}(y)) \rightarrow (x \neq y) \]

- DeMorgan’s Law for Quantifiers
- DeMorgan’s Law for Quantifiers
- DeMorgan’s Law
- Definition of \( \neq \)
- Law of Implication
Theorems and Proofs
Theorems and Proofs

- **Definition:**
- A **theorem** is a statement that can be shown to be true.

- A **proof** is a valid argument that establishes a statement to be true.
Theorems and Proofs

- Examples of theorems include...
  - Given a right triangle with side lengths \(a, b\) and hypotenuse \(c\), \(a^2 + b^2 = c^2\)
  - There are infinitely many prime numbers.
  - There exists a problem that cannot be solved by a program.
Integer

- **Definition:**
- An integer is a real number with no fractional part.

**e.g.** $-17, 0, 1, 53$
Odd and Even

Definitions:

- An integer $x$ is even iff there exists an integer $k$ such that $x = 2k$.
  \[ \text{Even}(x) := \exists k \,(x = 2k) \]

- An integer $x$ is odd iff there exists an integer $k$ such that $x = 2k + 1$.
  \[ \text{Odd}(x) := \exists k \,(x = 2k + 1) \]
Arbitrary Variables
Proof Strategy: Direct Proof
Direct Proof

- Direct proof is one strategy for proving statements of the form
- $\forall x [P(x) \rightarrow Q(x)]$
Our First Direct Proof

- **Prove:** “For all integers $x$, if $x$ is even, then $x^2$ is even.”

- What’s the claim in logic?

- How would we prove this claim?
- We’ll see how to prove it formally in a minute; for now, just try to convince each other this statement is true.
Our First Direct Proof

• Prove: “For all integers \( x \), if \( x \) is even, then \( x^2 \) is even.” \( \forall x (\text{Even}(x) \rightarrow \text{Even}(x^2)) \)

Definitions
Even\((x) := \exists k (x = 2k)\)
Arbitrary

- An “arbitrary” variable is one that
  - is part of the domain of discourse (or some sub-domain you pick).
  - You know nothing else about.

- EVERY element of the domain could be plugged into that arbitrary variable. And everything else you say in the proof will follow.

- An arbitrary variable is exactly what you need to convince us of a $\forall$.

- If you want to prove a for-all you must explicitly tell us the variable is arbitrary.
  - Your reader doesn’t know what you’re doing otherwise.
Our First Direct Proof

• Prove: “For all integers $x$, if $x$ is even, then $x^2$ is even.” $\forall x (\text{Even}(x) \rightarrow \text{Even}(x^2))$
• Proof: Let $x$ be an arbitrary integer. Suppose that $x$ is even.

Definitions
Even($x$) := $\exists k (x = 2k)$
Now What?

• Well….what does it mean to be even?

• $x = 2k$ for some integer $k$.

• Where do we need to end up?
  • $\text{Even}(x^2)$

Definitions
$\text{Even}(x) := \exists k (x = 2k)$
Our First Direct Proof

- Prove: “For all integers \( x \), if \( x \) is even, then \( x^2 \) is even.”  \( \forall x (\text{Even}(x) \rightarrow \text{Even}(x^2)) \)
- Proof: Let \( x \) be an arbitrary integer. Suppose that \( x \) is even.

  - So \( x^2 \) is even.
Our First Direct Proof

- Prove: “For all integers $x$, if $x$ is even, then $x^2$ is even.” $\forall x (\text{Even}(x) \rightarrow \text{Even}(x^2))$
- Proof: Let $x$ be an arbitrary integer. Suppose that $x$ is even.
- By definition of even, $x = 2k$ for some integer $k$.

- So $x^2$ is even.
Our First Direct Proof

Prove: “For all integers \( x \), if \( x \) is even, then \( x^2 \) is even.”  \( \forall x (\text{Even}(x) \rightarrow \text{Even}(x^2)) \)

Proof: Let \( x \) be an arbitrary integer. Suppose that \( x \) is even.

By definition of even, \( x = 2k \) for some integer \( k \).

Squaring both sides, we see that:
\[
    x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2
\]

Because \( k \) is an integer, \( 2k^2 \) is also an integer.

So \( x^2 \) is two times an integer.

Which is exactly the definition of even, so \( x^2 \) is even.

Since \( x \) was an arbitrary integer, we conclude that for all integers \( x \), if \( x \) is even then \( x^2 \) is also even.
Direct Proof Template

Declare an arbitrary variable for each \( \forall \).

Assume the left side of the implication.

Unroll the predicate definitions.

Manipulate towards the goal.

Reroll definitions into the right side of the implication.

Conclude that you have proved the claim.

Let \( x \) be an arbitrary integer.

Suppose that \( x \) is even.

Then by definition of even, there exists some integer \( k \) such that \( x = 2k \).

Squaring both sides, we see that:

\[
x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2
\]

Because \( k \) is an integer, then \( 2k^2 \) is also an integer. So \( x^2 \) is two times an integer.

So by definition of even, \( x^2 \) is even.

Since \( x \) was an arbitrary integer, we can conclude that for all integers \( x \), if \( x \) is even then \( x^2 \) is even.

Prove: \( \forall x (\text{Even}(x) \rightarrow \text{Even}(x^2)) \)
Direct Proof Steps

These are the usual steps. We’ll see different outlines in the future!!

- Introduction
  - Declare an arbitrary variable for each $\forall$ quantifier
  - Assume the left side of the implication
- Core of the proof
  - Unroll the predicate definitions
  - Manipulate towards the goal (using creativity, algebra, etc.)
  - Reroll definitions into the right side of the implication
- Conclude that you have proved the claim
Another Direct Proof

- Prove: “The product of two odd integers is odd.”

- What’s the claim in logic? \( \forall x \forall y \left( (\text{Odd}(x) \land \text{Odd}(y)) \rightarrow \text{Odd}(xy) \right) \)

- How would we prove this claim?
  Direct Proof. In particular, we’ll let \( x, y \) be arbitrary integers. We’ll suppose \( x, y \) are odd. We’ll show that \( x \cdot y \) is odd.
• Prove: “The product of two odd integers is odd.”
  \[ \forall x \forall y \left( (\text{Odd}(x) \land \text{Odd}(y)) \rightarrow \text{Odd}(xy) \right) \]

Let \( x \) and \( y \) be arbitrary integers. Suppose that \( x \) and \( y \) are odd. Then by definition of odd, there exists some integer \( k \) such that \( x = 2k + 1 \), and some integer \( j \) such that \( y = 2j + 1 \).

Then multiplying \( x \) and \( y \), we can see that:
  \[ xy = (2k + 1) \cdot (2j + 1) = 4kj + 2j + 2k + 1 = 2(2kj + j + k) + 1 \]

Since \( k, j \) are integers, \( 2kj + j + k \) is an integer. So by definition of odd, \( xy \) is odd. Since \( x, y \) were arbitrary, we have shown that the product of two odd integers is odd.
A note on Domain of Discourse

“The product of two odd integers is odd.”

Domain: Integers

Translation:
\( \forall x \forall y \left( (\text{Odd}(x) \land \text{Odd}(y)) \rightarrow \text{Odd}(xy) \right) \)

Proof Outline:
Let \( x \) and \( y \) be arbitrary integers.
Suppose \( x \) and \( y \) are odd.
Show \( xy \) is odd.

Domain: Odd Integers

Translation:
\( \forall x \forall y (\text{Odd}(xy)) \)

Proof Outline:
Let \( x \) and \( y \) be arbitrary odd integers.
Show \( xy \) is odd.
A note on Translation to Logic

- We first translate the claim to predicate logic because:
  - The translation makes it precise what we are proving
  - The translation hints at the structure of the proof
    e.g. for each $\forall$, introduce an arbitrary variable
- In practice, computer scientists identify the proof claim and structure without predicate logic translation
- Eventually we’ll stop asking you to translate to logic first
Square

- Definition:
- An integer \( x \) is \textit{square} iff there exists an integer \( k \) such that \( x = k^2 \).

- Square(\( x \)) := \exists k \ (x = k^2)
Yet Another Direct Proof

Prove: The product of two square integers is square.

What’s the claim in logic?

\[ \forall n \forall m \left( (\text{Square}(n) \land \text{Square}(m)) \rightarrow \text{Square}(nm) \right) \]

Prove this claim.
Yet Another Direct Proof

Prove: “The product of two square integers is square.”
\[ \forall n \forall m \left( (\text{Square}(n) \land \text{Square}(m)) \Rightarrow \text{Square}(nm) \right) \]

Let \( n \) and \( m \) be arbitrary integers. Suppose that \( n \) and \( m \) are square. Then by definition of square, \( n = k^2 \) for some integer \( k \), and \( m = j^2 \) for some integer \( j \).

Then multiplying \( n \) and \( m \), we can see:
\[ nm = k^2 \cdot j^2 = (kj)^2 \]

Since \( k \) and \( j \) are integers, \( kj \) is an integer. So by definition of square, \( nm \) is square. Since \( n \) and \( m \) were arbitrary, we have shown that the product of two square integers is square.

Definitions
\[
\text{Square}(x) := \exists k \ (x = k^2)
\]