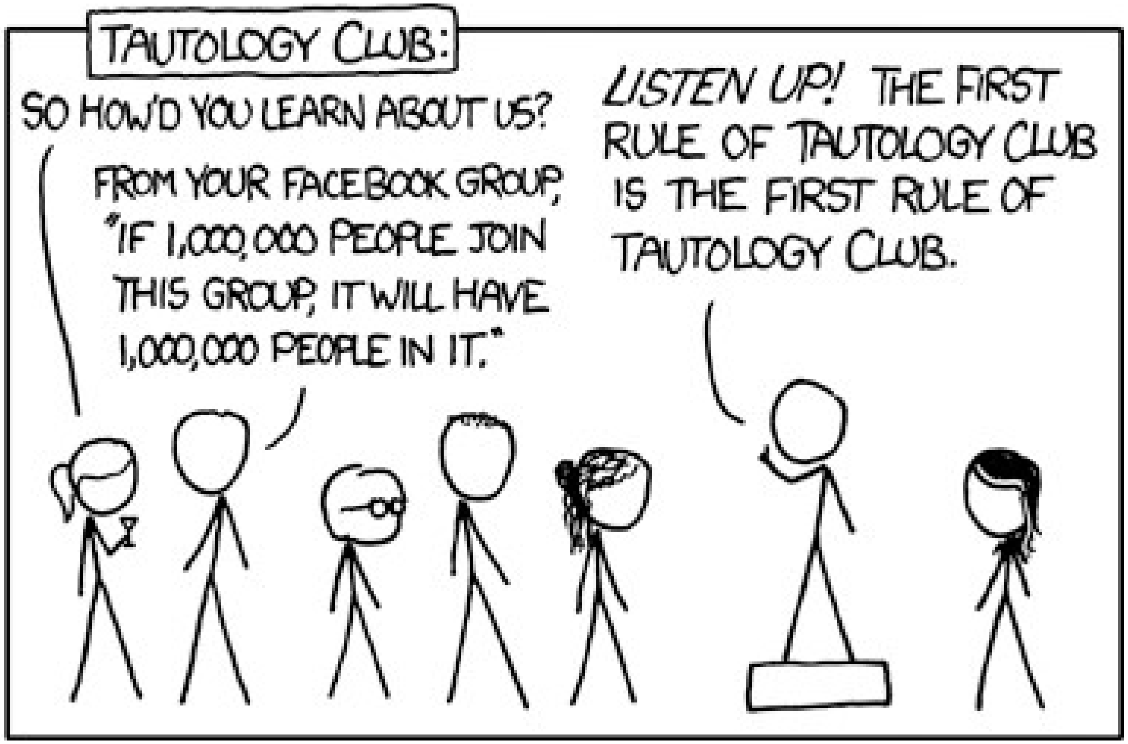


Formal Proofs, Predicate Logic

CSE 311
Lecture 4

Announcements

- Homework 1 is due this Friday
- Section tomorrow
  - If you are not in person, your worksheet is due emailed to your TAs by 8PM.

Inference Proofs

Recall: Propositional Inference Rules

Two inference rules per binary connective, one to eliminate it and one to introduce it

Elim ∧

$$\frac{A \wedge B}{\therefore A, B}$$

Intro ∧

$$\frac{A \,;\, B}{\therefore A \wedge B}$$

Elim ∨

$$\frac{A \vee B \,;\, \neg A}{\therefore B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

Modus Ponens

$$\frac{A \,;\, A \rightarrow B}{\therefore B}$$

Direct Proof

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

Proofs

Prove that $\neg r$ follows from $p \land s$, $q \to \neg r$, and $\neg s \lor q$.

1.	$p \land s$	Given
2.	$q \to \neg r$	Given
3.	$\neg s \lor q$	Given
4.	s	$\land$ Elim: 1
5.	$\neg\neg s$	Double Negation: 4
6.	q	$\lor$ Elim: 3, 5
7.	$\neg r$	MP: 2, 6

Important: Applications of Inference Rules

- You can use equivalences to make substitutions

  of any sub-formula.

  e.g. $(p \rightarrow r) \vee q \equiv (\neg p \vee r) \vee q$

- Inference rules only can be applied to whole formulas (not correct otherwise).

  e.g. 1. $p \rightarrow r$ given

  ~~2. $(p \vee q) \rightarrow r$ intro $\vee$ from 1.~~

  Does not follow! e.g . p=F, q=T, r=F

Recall: Propositional Inference Rules

Two inference rules per binary connective, one to eliminate it and one to introduce it

Elim ∧

$$\frac{A \wedge B}{\therefore A, B}$$

Intro ∧

$$\frac{A \,;\, B}{\therefore A \wedge B}$$

Elim ∨

$$\frac{A \vee B \,;\, \neg A}{\therefore B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

Modus Ponens

$$\frac{A \,;\, A \rightarrow B}{\therefore B}$$

Direct Proof

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

Recall: New Perspective

Rather than comparing A and B as columns,

zooming in on just the rows where A is true:

p	***q***	**A**	**B**
T	T	T	T
T	F	T	T
F	T	F	
F	F	F	

Given that A is true, we see that B is also true.

A ® B

Recall: New Perspective

Rather than comparing A and B as columns,

zooming in on just the rows where B is true:

p	q	**A**	**B**	**A → B**
T	T	T	T	T
T	F	T	T	T
F	T	F	T	T
F	F	F	F	T

When we zoom out, what have we proven?

$$(A \to B) \equiv \mathbf{T}$$

Recall: Propositional Inference Rules

Two inference rules per binary connective, one to eliminate it and one to introduce it

Elim ∧
$$\frac{A \wedge B}{\therefore A, B}$$

Intro ∧
$$\frac{A \,;\, B}{\therefore A \wedge B}$$

Elim ∨
$$\frac{A \vee B \,;\, \neg A}{\therefore B}$$

Intro ∨
$$\frac{A}{\therefore A \vee B, B \vee A}$$

Modus Ponens
$$\frac{A \,;\, A \rightarrow B}{\therefore B}$$

Direct Proof
$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

Not like other rules

To Prove An Implication: $A \rightarrow B$

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

- We use the direct proof rule
- The "pre-requisite" $A \Rightarrow B$ for the direct proof rule is a proof that "Assuming A, we can prove B."
- The direct proof rule:
  If you have such a proof, then you can conclude that $A \rightarrow B$ is true

Proofs using the direct proof rule

Show that $p \rightarrow r$ follows from q and $(p \wedge q) \rightarrow r$

1. q Given
2. $(p \wedge q) \rightarrow r$ Given

This is a proof of $p \rightarrow r$

3.1. p Assumption
3.2.
3.3. r **??**

If we know p is true… Then, we've shown r is true

3. $p \rightarrow r$ Direct Proof

Proofs using the direct proof rule

Show that $p \to r$ follows from q and $(p \wedge q) \to r$

1. q Given
2. $(p \wedge q) \to r$ Given

    3.1. p Assumption

    3.2. $p \wedge q$ Intro $\wedge$: **1, 3.1**

    3.3. r **MP: 2, 3.2**

3. $p \to r$ Direct Proof

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

There MUST be an application of the Direct Proof Rule (or an equivalence) to prove this implication.

Where do we start? We have no givens…

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

1.1.	$p \wedge q$	**Assumption**
1.9.	$p \vee q$	**??**
1.	$(p \wedge q) \rightarrow (p \vee q)$	**Direct Proof**

Example

Prove: $(p \land q) \rightarrow (p \lor q)$

1.1.	$p \land q$	**Assumption**
1.2.	p	**Elim $\land$: 1.1**
1.3.	$p \lor q$	**Intro $\lor$: 1.2**
1.	$(p \land q) \rightarrow (p \lor q)$	**Direct Proof**

One General Proof Strategy

1. Look at the rules for introducing connectives to see how you would build up the formula you want to prove from pieces of what is given

2. Use the rules for eliminating connectives to break down the given formulas so that you get the pieces you need to do 1.

3. Write the proof beginning with what you figured out for 2 followed by 1.

Example

Prove: $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

Example

Prove: $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

1.1. $(p \rightarrow q) \wedge (q \rightarrow r)$ **Assumption**

1.? $p \rightarrow r$

1. $\big((p \rightarrow q) \wedge (q \rightarrow r)\big) \rightarrow (p \rightarrow r)$ **Direct Proof**

Example

Prove: $((p \to q) \land (q \to r)) \to (p \to r)$

1.1.	$(p \to q) \land (q \to r)$	**Assumption**
1.2.	$p \to q$	**$\land$ Elim: 1.1**
1.3.	$q \to r$	**$\land$ Elim: 1.1**
1.?	$p \to r$	

1. $((p \to q) \land (q \to r)) \to (p \to r)$ **Direct Proof**

Example

Prove: $((p \to q) \land (q \to r)) \to (p \to r)$

1.1. $(p \to q) \land (q \to r)$ **Assumption**

1.2. $p \to q$ **∧ Elim: 1.1**

1.3. $q \to r$ **∧ Elim: 1.1**

1.4.1. p **Assumption**

1.4.? r

1.4. $p \to r$ **Direct Proof**

1. $((p \to q) \land (q \to r)) \to (p \to r)$ **Direct Proof**

Example

Prove: $((p \to q) \land (q \to r)) \to (p \to r)$

1.1. $(p \to q) \land (q \to r)$ **Assumption**

1.2. $p \to q$ **∧ Elim: 1.1**

1.3. $q \to r$ **∧ Elim: 1.1**

 1.4.1. p **Assumption**

 1.4.2. q **MP: 1.2, 1.4.1**

 1.4.3. r **MP: 1.3, 1.4.2**

1.4. $p \to r$ **Direct Proof**

1. $((p \to q) \land (q \to r)) \to (p \to r)$ **Direct Proof**

Minimal Rules for Propositional Logic

Can get away with just these:

Elim ∧

$$\frac{A \wedge B}{\therefore A,\ B}$$

Intro ∧

$$\frac{A\ ;\ B}{\therefore A \wedge B}$$

Elim ∨

$$\frac{A \vee B\ ;\ \neg A}{\therefore B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B,\ B \vee A}$$

Modus Ponens

$$\frac{A\ ;\ A \rightarrow B}{\therefore\ B}$$

Direct Proof

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

Excluded Middle

$$\frac{}{\therefore A \vee \neg A}$$

More Rules for Propositional Logic

More rules makes proofs easier

Elim ∧

$$\frac{A \wedge B}{\therefore A, B}$$

Intro ∧

$$\frac{A ; B}{\therefore A \wedge B}$$

Elim ∨

$$\frac{A \vee B ; \neg A}{\therefore B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

Modus Ponens

$$\frac{A ; A \rightarrow B}{\therefore B}$$

Direct Proof

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

Excluded Middle

$$\frac{}{\therefore A \vee \neg A}$$

Equivalent

$$\frac{A \equiv B ; B}{\therefore A}$$

Example Formal Proof:

Show that $\neg p$ follows from $\neg(\neg r \vee t), \neg q \vee \neg s,$ and $(p \rightarrow q) \wedge (r \rightarrow s)$

1. $\neg(\neg r \vee t)$	Given
2. $\neg q \vee \neg s$	Given
3. $(p \rightarrow q) \wedge (r \rightarrow s)$	Given
4. $r \wedge t$	Equivalent 1. (double negation and demorgans law)
5. $r \rightarrow s$	Elim And 3.
6. r	Elim And 4.
7. s	MP, 5, 6.
8. $\neg\neg s$	Equivalent 7 (double negation).
9. $\neg q$	Elim or, 2, 8.
10. $p \rightarrow q$	Elim And, 3.
11. $\neg q \rightarrow \neg p$	Contrapositive, 10
12. $\neg p$	MP, 9, 11.

Predicate Logic

Motivation

- Often we will work with statements of the form:
  - If $x > 10$, then $x^2 > 100$.
  - If x is even, x^2 is even.

- Can you translate these to propositional logic?
- No. We need a function that is true or false depending on the value of x.

Motivation

- Propositional Logic
- Lets us break down complex true or false statements into atomic parts joined by connectives.

- Predicate Logic
- Lets us analyze complex true or false statements that are functions of some underlying objects.

Predicate Logic

- 3 Parts

1. Predicate
2. Domain of Discourse
3. Quantifiers

Predicate

- Definition:
- A predicate is a function that outputs true or false.

- $\text{Cat}(x) \coloneqq x$ is a cat
- $\text{Even}(x) \coloneqq x$ is even
- $\text{LessThan}(x, y) \coloneqq x < y$
- $\text{Sum}(x, y, z) \coloneqq x + y = z$
- $\text{IsRaining} \coloneqq$ It is raining outside

Analogy

- Propositions were like Boolean variables.
- `boolean itIsRaining = true`

- Predicates are like functions that return Boolean values.
- `public boolean Even(int x) {…}`

Predicate Translation: Example

- x is prime or x^2 is odd, or $x = 2$

- $\text{Prime}(a) \coloneqq a$ is Prime
- $\text{Odd}(a) \coloneqq a$ is Odd
- $\text{Equals}(a, b) \coloneqq \; a = b$

$$\text{Prime}(x) \vee \text{Odd}(x^2) \vee \text{Equals}(x, 2)$$

Domain of Discourse

- Definition:
- The domain of discourse for a predicate is the set of possible inputs.

- $\mathrm{Cat}(x)$ – Possible domains include mammals, animals, cats.

- $\mathrm{LessThan}(x, y)$ – Possible domains include numbers, integers.

Domain of Discourse: Example

- What's a possible domain of discourse for these predicates?

1. $\mathrm{Prime}(x) \coloneqq x$ is prime
   Positive Integers, Integers, all Numbers

2. $\mathrm{Equals}(x, y) \coloneqq x$ and y are the same object
   Integers, all Numbers, all People, all Mammals

3. $\mathrm{EnrolledIn}(x, y) \coloneqq x$ is enrolled in course y
   Students and Courses

Quantifiers: Motivation

- We tend to use variables for two reasons:

1. The statement is true for every x.
   For every integer x, if x is even then x^2 is even.

2. There is some x for which the statement is true.
   There is some problem x that computers cannot solve.

Quantifiers

- $\forall x\ \mathrm{P}(x)$

- $\forall$ is called the Universal Quantifier.
- Read out loud as "for all x, P of x".
- $\forall x\ \mathrm{P}(x)$ means for every x in the domain, $\mathrm{P}(x)$ is true.

- $\exists x\ \mathrm{P}(x)$

- $\exists$ is called the Existensial Quantifier.
- Read out loud as "there exists x, P of x".
- $\exists x\ \mathrm{P}(x)$ means there is some x in the domain for which $\mathrm{P}(x)$ is true.

Predicate Logic Summary

- <u>3 Parts</u>

1. Predicate – Function that outputs true or false.

2. Domain of Discourse – Possible inputs to a predicate statement.

3. Quantifiers – A statement about when a predicate is true: ∀ or ∃

Predicate Logic Translation

English to Logic

Domain of Discourse
Integers

Predicate Definitions
$\text{Even}(x) \coloneqq x$ is even
$\text{Equals}(x, y) \coloneqq x = y$
$\text{LessThan}(x, y) \coloneqq x < y$

Translate to predicate logic. Then evaluate if the statement is $\mathbf{T}$ or $\mathbf{F}$.

- For every integer x, if x is even, then $x = 2$.
- $\forall x\big(\text{Even}(x) \to \text{Equals}(x, 2)\big)$
- False, e.g. $x = 4$

- There are integers x and y such that $x < y$.
- $\exists x \exists y\big(\text{LessThan}(x, y)\big)$
- True, e.g. $x = 2, y = 3$

Logic to English

Domain of Discourse
Integers

Predicate Definitions
$\text{Even}(x) := x$ is even
$\text{LessThan}(x, y) := x < y$
$\text{Odd}(x) := x$ is odd

Translate to English. Then evaluate if the statement is $\mathbf{T}$ or $\mathbf{F}$.

- $\exists x\big(\text{Odd}(x) \wedge \text{LessThan}(x, 5)\big)$
- There is an odd integer less than 5.
- True, e.g. $x = 3$.

- $\forall y\big(\text{Even}(y) \wedge \text{Odd}(y)\big)$
- All integers are even and odd.
- False.

Examples

Domain of Discourse
Integers

Predicate Definitions
$\text{Even}(x) \coloneqq x$ is even $\text{Greater}(x, y) \coloneqq x > y$
$\text{Odd}(x) \coloneqq x$ is odd $\text{Prime}(x) \coloneqq x$ is prime

Translate to English. Then evaluate if the statement is T or F.

$\exists x\ \text{Even}(x)$	There is an even integer.	T
$\forall x\ \text{Odd}(x)$	All integers are odd.	F
$\forall x\ (\text{Even}(x) \vee \text{Odd}(x))$	All integers are even or odd.	T
$\exists x\ (\text{Even}(x) \wedge \text{Odd}(x))$	There's an integer that's even and odd.	F
$\forall x\ \text{Greater}(x + 1, x)$	For all integers x, $x + 1 > x$	T
$\exists x(\text{Even}(x) \wedge \text{Prime}(x))$	There is an integer that's even and prime.	T

Domain Restriction

Domain Restriction

- Definition:
- Domain restriction is the technique of limiting our domain of discourse to a smaller set of objects.

Domain Restriction

Domain of Discourse
Animals

Predicate Definitions
$\mathrm{Cat}(x) \coloneqq x$ is a cat
$\mathrm{Blue}(x) \coloneqq x$ is blue

- All cats are blue.
- $\forall x\left(\mathrm{Cat}(x) \rightarrow \mathrm{Blue}(x)\right)$

- There is a blue cat.
- $\exists x\left(\mathrm{Cat}(x) \land \mathrm{Blue}(x)\right)$

Domain Restriction

Domain of Discourse
Animals

Predicate Definitions
$\text{Cat}(x) \coloneqq x$ is a cat
$\text{Blue}(x) \coloneqq x$ is blue

- $\forall x\big(\text{Cat}(x) \rightarrow \text{Blue}(x)\big)$ vs. $\forall x\big(\text{Cat}(x) \wedge \text{Blue}(x)\big)$
  - All cats are blue.

  All animals are blue cats.

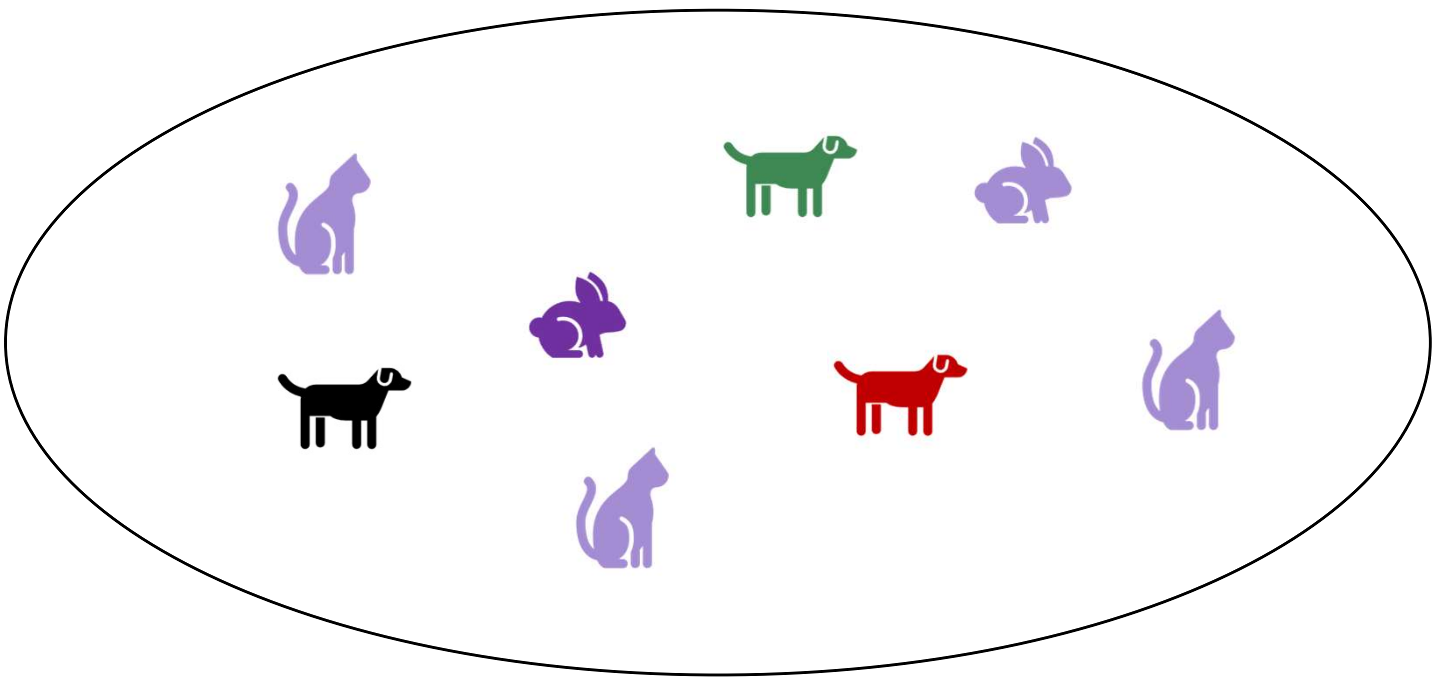
- $\exists x\big(\text{Cat}(x) \rightarrow \text{Blue}(x)\big)$ vs. $\exists x\big(\text{Cat}(x) \wedge \text{Blue}(x)\big)$
  - There is an animal that is not a cat or is blue

  There is a blue cat.

Tip: Avoid → under ∃ in most situations

∀ and vacuous truth

- $\forall x\big(\mathrm{Cat}(x) \rightarrow \mathrm{Blue}(x)\big)$ means "all cats are blue".
- One way to check if $\forall x\ \mathrm{P}(x)$ is true is to "loop" over every element of the domain and check that $\mathrm{P}(x)$ is true.



Domain Restriction

- Translations often sound more natural if we:
  1. Notice domain restriction patterns.
  2. Avoid using variables when we can.
  3. Drop the “for all” or “there exists” when we can.

- For example:
- $\forall x(\text{Cat}(x) \rightarrow \text{Blue}(x))$
- ❌ For all animals x, if x is a cat then x is blue.
- ✔ All cats are blue.

Domain Restriction

<u>Domain of Discourse</u>
Food

<u>Predicate Definitions</u>
$\text{Fruit}(x) \coloneqq x$ is a fruit
$\text{Tasty}(x) \coloneqq x$ is tasty
$\text{Ripe}(x) \coloneqq x$ is ripe

- Translate these sentences using a **natural-sounding** translation.
- $\exists x \big(\text{Fruit}(x) \land \text{Tasty}(x)\big)$
- There is a tasty fruit. OR Some fruits are tasty.

- $\forall x \Big(\big(\text{Fruit}(x) \land \neg\text{Ripe}(x)\big) \rightarrow \neg\text{Tasty}(x)\Big)$
- All fruits that aren't ripe aren't tasty.

Quantifier Scope

- $\exists x \left(\mathrm{P}(x) \wedge \mathrm{Q}(x)\right)$ vs. $\exists x\ \mathrm{P}(x) \wedge \exists x\ \mathrm{Q}(x)$

Could be different x's

For example

Domain of Discourse: Integers

$\mathrm{P}(x) \coloneqq x$ is odd

$\mathrm{Q}(x) \coloneqq x$ is even

Section Tomorrow

Practice with equivalences & translating

Quantifiers can be nested!

There is a person that all people love.

$\exists x \forall y (\text{Love}(y, x))$