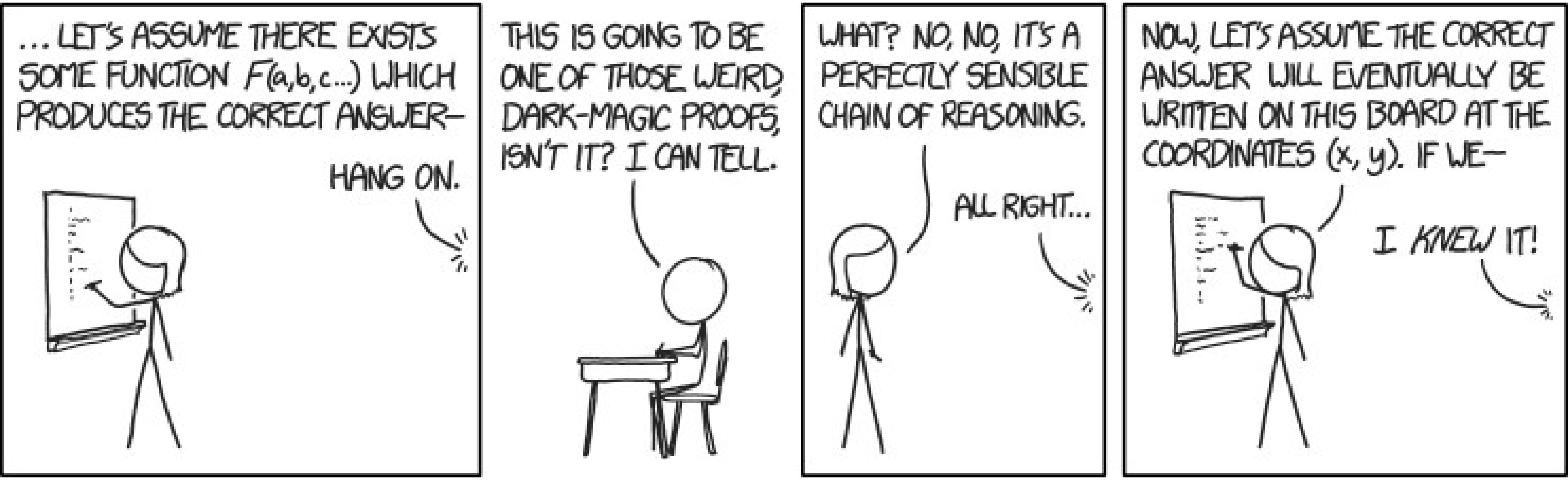



# Equivalences, Formal Proofs

CSE 311
Lecture 2

# Announcements

- Homework 1 is due this Friday
- Office hours start today!

# Equivalence Rules:

For every propositions $p, q, r$ the following hold:

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$
- **DeMorgan's Laws**
  - $\neg(p \vee q) \equiv \neg p \wedge \neg q$
  - $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- **Double Negation**

  $\neg\neg p \equiv p$
- **Law of Implication**

  $p \rightarrow q \equiv \neg p \vee q$
- **Contrapositive**

  $p \rightarrow q \equiv \neg q \rightarrow \neg p$

# Logical Equivalence Examples

## Ex 1: Prove $(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q) \equiv p \rightarrow q$

| | |
|---|---|
| $(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q) \equiv \big((p \vee \neg p) \wedge q\big) \vee (\neg p \wedge \neg q)$ | Distributivity |
| $\equiv (\mathrm{T} \wedge q) \vee (\neg p \wedge \neg q)$ | Negation |
| $\equiv q \vee (\neg p \wedge \neg q)$ | Identity |
| $\equiv (q \vee \neg p) \wedge (q \vee \neg q)$ | Distributivity |
| $\equiv (q \vee \neg p) \wedge \mathrm{T}$ | Negation |
| $\equiv q \vee \neg p$ | Identity |
| $\equiv \neg p \vee q$ | Commutativity |
| $\equiv p \rightarrow q$ | L.O.I |

# Caveat 1: Associativity & Commutativity

- Show that $(p \vee q) \vee (r \vee s) \equiv r \vee (q \vee s) \vee p$, following rules exactly.

| | |
|---|---|
| $(p \vee q) \vee (r \vee s) \equiv p \vee \big(q \vee (r \vee s)\big)$ | Associativity |
| $\equiv p \vee \big(q \vee (s \vee r)\big)$ | Commutativity |
| $\equiv p \vee \big((q \vee s) \vee r\big)$ | Associativity |
| $\equiv \big((q \vee s) \vee r\big) \vee p$ | Commutativity |
| $\equiv \big(r \vee (q \vee s)\big) \vee p$ | Commutativity |
| $\equiv r \vee (q \vee s) \vee p$ | Order of Operations |

# Caveat 1: Associativity & Commutativity

- Show that $(p \vee q) \vee (r \vee s) \equiv r \vee (q \vee s) \vee p$.
- We will allow abbreviated associativity & commutativity steps.

- $(p \vee q) \vee (r \vee s) \equiv r \vee (q \vee s) \vee p$ Associativity & Commutativity

# Caveat 1: Associativity & Commutativity

- Show that $\neg p \lor p \equiv \mathrm{T}$.

Showing all steps:

$\neg p \lor p \equiv p \lor \neg p$ Commutativity

$\equiv \mathrm{T}$ Negation

What we allow:

$\neg p \lor p \equiv \mathrm{T}$ Negation

# Caveat 2: Applying a rule twice

- Expand $(p \to q) \vee (q \to r)$ using the Law of Implication.

- Showing all steps:

$$
\begin{aligned}
(p \to q) \vee (q \to r) &\equiv (\neg p \vee q) \vee (q \to r) && \text{Law of Implication} \\
&\equiv (\neg p \vee q) \vee (\neg q \vee r) && \text{Law of Implication}
\end{aligned}
$$

- What we allow:

$$(p \to q) \vee (q \to r) \equiv (\neg p \vee q) \vee (\neg q \vee r) \qquad \text{Law of Implication (x2)}$$

# Caveat 3: Applying rules to any proposition

- We can apply equivalence rules to any proposition.
- $(p \vee q \wedge r) \vee (p \vee q \wedge r) \equiv p \vee q \wedge r$ Idempotency
- $\neg\neg(r \rightarrow \neg q) \equiv r \rightarrow \neg q$ Double Negation
- $\neg\big((p \vee q) \wedge s\big) \equiv \neg(p \vee q) \vee \neg s$ DeMorgan's Law

# Ex 2: Prove $\neg p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \vee r)$

| | |
|---|---|
| $\neg p \rightarrow (q \rightarrow r) \equiv \neg\neg p \vee (\neg q \rightarrow r)$ | Law of Implication |
| $\equiv \neg\neg p \vee (\neg q \vee r)$ | Law of Implication |
| $\equiv p \vee (\neg q \vee r)$ | Double Negation |
| $\equiv \neg q \vee (p \vee r)$ | Assoc, Com |
| $\equiv q \rightarrow (p \vee r)$ | Law of Implication |

# Boolean Algebra & Circuits

Note: We do not have time to fully cover this, so this concept will only be one question on your homework 2, and not on any of the exams except as potentially a multiple-choice problem.

# Notation

- Logic is fundamental
  - Computer scientists use it in programs
  - Mathematicians use it in proofs
  - Engineers use it in hardware
  - Philosophers use it in arguments
- Consequently, everyone has their own notation

# Boolean Algebra

- Another notation for logic. Preferred by some because it's compact.

| Term | Propositional Logic | Boolean Algebra |
|---|---|---|
| or | $\vee$ | $+$ |
| and | $\wedge$ | $\cdot$ |
| not | $\neg$ | $'$ |
| True | $\mathbf{T}$ | $\mathbf{1}$ |
| False | $\mathbf{F}$ | $\mathbf{0}$ |

- $(p \wedge q \wedge r) \vee s \vee \neg t$ in Boolean Algebra is $pqr + s + t'$.

# Boolean Algebra: Example Translation

Translate the following into Boolean Algebra notation:

$$p \vee \neg(\neg q \wedge r)$$

$$\downarrow$$

$$p + (q' \cdot r)'$$

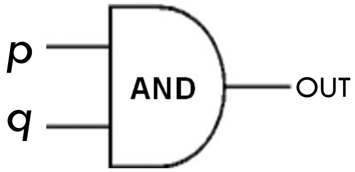
# Digital Circuits

- Computing with Logic
  - $\mathbf{T}$ corresponds to $\mathbf{1}$, or high voltage
  - $\mathbf{F}$ corresponds to $\mathbf{0}$, or low voltage

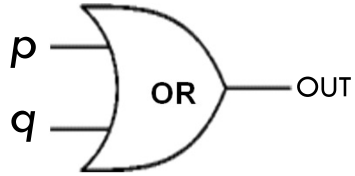
- Gates
  - Gates take inputs and produce outputs

# Digital Circuits – Gates

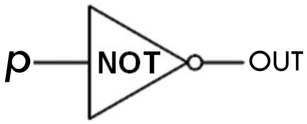
AND Gate




OR Gate

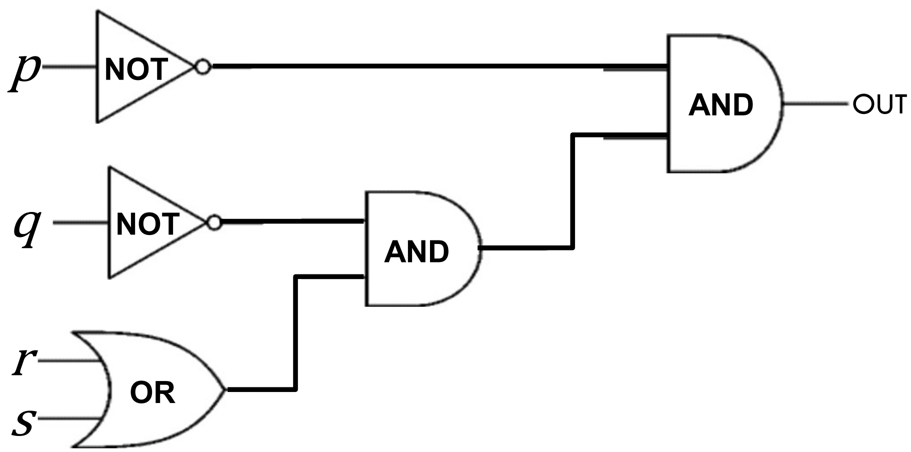


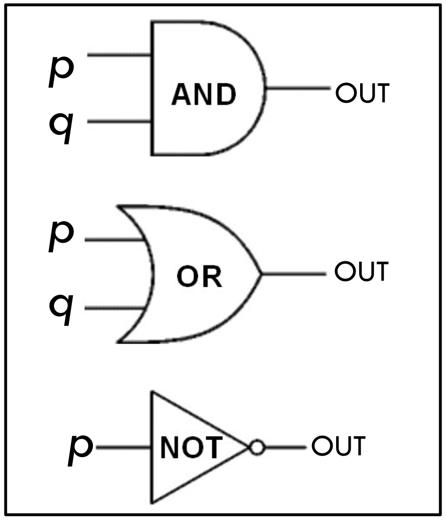

NOT Gate



# Digital Circuits – Example

- Write $\neg p \wedge (\neg q \wedge (r \vee s))$ as a circuit.



# Inference Proofs

# Logical Inference

- So far, we've considered:
  - how to understand and *express* things using propositional and predicate logic
  - how to *compute* using Boolean (propositional) logic
  - how to show that different ways of expressing or computing them are *equivalent* to each other

- Logic also has methods that let us *infer* implied properties from ones that we know
  - equivalence is a small part of this

# New Perspective

Rather than comparing A and B as columns, zooming in on just the rows where A is true:

| ***p*** | ***q*** | **A(*p*,*q*)** | **B(*p*,*q*)** |
|---|---|---|---|
| T | T | T | |
| T | F | T | |
| F | T | F | |
| F | F | F | |

# New Perspective

Rather than comparing A and B as columns, zooming in on just the rows where A is true:

| *p* | *q* | **A(*p*,*q*)** | **B(*p*,*q*)** |
|---|---|---|---|
| T | T | T | T |
| T | F | T | T |
| F | T | F | |
| F | F | F | |

Given that A is true, we see that B is also true.

A ® B

# New Perspective

Rather than comparing A and B as columns,

zooming in on just the rows where A is true:

| ***p*** | ***q*** | **A(*p*,*q*)** | **B(*p*,*q*)** |
|---|---|---|---|
| T | T | T | T |
| T | F | T | T |
| F | T | F | ? |
| F | F | F | ? |

When we zoom out, what have we proven?

# New Perspective

Rather than comparing A and B as columns,

zooming in on just the rows where A is true:

| $p$ | $q$ | A($p$,$q$) | B($p$,$q$) | A → B |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | T | T | T |
| F | T | F | T | T |
| F | F | F | F | T |

When we zoom out, what have we proven?

$$(A \rightarrow B) \equiv \mathbf{T}$$

# New Perspective

Equivalences

$A \equiv B$ and $(A \leftrightarrow B) \equiv T$ are the same

Inference

A ® B and $(A \rightarrow B) \equiv T$ are the same

Can do the inference by zooming in to the rows where $A$ is true

- that is, we <u>assume</u> that $A$ is true

# Applications of Logical Inference

- Software Engineering
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- Artificial Intelligence
  - Automated reasoning
- Algorithm design and analysis
  - e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

# Proofs

- Start with given facts (hypotheses)
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

# An inference rule: *Modus Ponens*

- If A and $A \rightarrow B$ are both true, then B must be true

- Write this rule as $$\frac{A\ ;\ A \rightarrow B}{\therefore\ B}$$

- Given:
  - If it is Wednesday, then you have a 311 class today.
  - It is Wednesday.

- Therefore, by Modus Ponens:
  - You have a 311 class today.

# My First Proof!

Show that $r$ follows from $p$, $p \rightarrow q$, and $q \rightarrow r$

1. $p$ Given
2. $p \rightarrow q$ Given
3. $q \rightarrow r$ Given
4. 
5. 

**Modus Ponens** $\frac{A\ ;\ A \rightarrow B}{\therefore\ B}$

# My First Proof!

Show that $r$ follows from $p$, $p \to q$, and $q \to r$

| | | |
|---|---|---|
| 1. | $p$ | Given |
| 2. | $p \to q$ | Given |
| 3. | $q \to r$ | Given |
| 4. | $q$ | MP: 1, 2 |
| 5. | $r$ | MP: 3, 4 |

**Modus Ponens** $\dfrac{A\ ;\ A \to B}{\therefore\ B}$

# Proofs can use equivalences too

Show that $\neg p$ follows from $p \rightarrow q$ and $\neg q$

| | | |
|---|---|---|
| 1. | $p \rightarrow q$ | Given |
| 2. | $\neg q$ | Given |
| 3. | | |
| 4. | $\neg p$ | |

Modus Ponens $\dfrac{A\ ;\ A \rightarrow B}{\therefore\ B}$

# Proofs can use equivalences too

Show that $\neg p$ follows from $p \rightarrow q$ and $\neg q$

| | | |
|---|---|---|
| **1.** | $p \rightarrow q$ | **Given** |
| **2.** | $\neg q$ | **Given** |
| **3.** | $\neg q \rightarrow \neg p$ | **Contrapositive: 1** |
| **4.** | $\neg p$ | **MP: 2, 3** |

**Modus Ponens** $\dfrac{A\ ;\ A \rightarrow B}{\therefore\ B}$

# Inference Rules

If A is true and B is true ....

Requirements: $\dfrac{A \;;\; B}{}$

Conclusions: $\therefore C \;,\; D$

Then, C must be true

Then D must be true

Example (Modus Ponens):

$$\frac{A \;;\; A \rightarrow B}{\therefore \quad B}$$

If I have A and A $\rightarrow$ B both true,
Then B must be true.

# Axioms: Special inference rules

**If I have nothing...**

**Requirements:** ______

**Conclusions:** $\therefore$ C , D

**Then, C must be true**

**Then D must be true**

**Example (Excluded Middle):**

$$\frac{}{\therefore\ A \vee \neg A}$$

**$A \vee \neg A$ must be true.**

# Simple Propositional Inference Rules

Two inference rules per binary connective,
one to eliminate it and one to introduce it

Elim ∧
$$\frac{A \land B}{\therefore A,\ B}$$

Intro ∧
$$\frac{A\ ;\ B}{\therefore A \land B}$$

Elim ∨
$$\frac{A \lor B\ ;\ \neg A}{\therefore B}$$

Intro ∨
$$\frac{A}{\therefore A \lor B,\ B \lor A}$$

Modus Ponens
$$\frac{A\ ;\ A \rightarrow B}{\therefore B}$$

Direct Proof
$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

# Proofs

Show that r follows from p, $p \rightarrow q$ and $(p \wedge q) \rightarrow r$

**How To Start:**

**We have givens, find the ones that go together and use them. Now, treat new things as givens, and repeat.**

$$\frac{A \,;\, A \rightarrow B}{\therefore\ B}$$

$$\frac{A \wedge B}{\therefore\ A, B}$$

$$\frac{A \,;\, B}{\therefore\ A \wedge B}$$

# Proofs

Show that $r$ follows from $p, p \to q$, and $p \wedge q \to r$

**Two visuals of the same proof. We will use the top one, but if the bottom one helps you think about it, that's great!**

| | | |
|---|---|---|
| 1. | $p$ | Given |
| 2. | $p \to q$ | Given |
| 3. | $q$ | MP: 1, 2 |
| 4. | $p \wedge q$ | Intro $\wedge$: 1, 3 |
| 5. | $p \wedge q \to r$ | Given |
| 6. | $r$ | MP: 4, 5 |

$$\dfrac{\dfrac{p \quad ; \quad \dfrac{p \quad ; \quad p \to q}{q}\,\text{MP}}{p \wedge q}\,\text{Intro } \wedge \quad ; \quad p \wedge q \to r}{r}\,\text{MP}$$

# Proofs

**Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.**

| | | |
|---|---|---|
| 1. | $p \wedge s$ | Given |
| 2. | $q \rightarrow \neg r$ | Given |
| 3. | $\neg s \vee q$ | Given |
| 20. | $\neg r$ | ? |

**First: Write down givens and goal**

**Idea: Work backwards!**

# Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

| | | |
|---|---|---|
| 1. | $p \wedge s$ | Given |
| 2. | $q \rightarrow \neg r$ | Given |
| 3. | $\neg s \vee q$ | Given |
| 20. | $\neg r$ | MP: 2, ? |

<u>**Idea: Work backwards!**</u>

We want to eventually get $\neg r$. How?

- We can use $q \rightarrow \neg r$ to get there.
- The justification between 2 and 20 looks like "elim $\rightarrow$" which is MP.

# Proofs

**Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.**

| | | |
|---|---|---|
| 1. | $p \wedge s$ | Given |
| 2. | $q \rightarrow \neg r$ | Given |
| 3. | $\neg s \vee q$ | Given |
| 19. | $q$ | ? |
| 20. | $\neg r$ | MP: 2, 19 |

**Idea: Work backwards!**

We want to eventually get $\neg r$. How?

- Now, we have a new "hole"
- We need to prove $q$...
  - Notice that at this point, if we prove $q$, we've proven $\neg r$...

# Proofs

**Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.**

| | | |
|---|---|---|
| 1. | $p \wedge s$ | Given |
| 2. | $q \rightarrow \neg r$ | Given |
| 3. | $\neg s \vee q$ | Given |
| 19. | $q$ | ? |
| 20. | $\neg r$ | MP: 2, 19 |

This looks like or-elimination.

Elim ∨ $\dfrac{A \vee B\ ;\ \neg A}{\therefore B}$

# Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

| | | |
|---|---|---|
| 1. | $p \wedge s$ | Given |
| 2. | $q \rightarrow \neg r$ | Given |
| 3. | $\neg s \vee q$ | Given |
| | | |
| 18. | $\neg\neg s$ | ? |
| 19. | $q$ | $\vee$ Elim: 3, 18 |
| 20. | $\neg r$ | MP: 2, 19 |

$\neg\neg s$ doesn't show up in the givens but $s$ does and we can use equivalences

# Proofs

**Prove that $\neg r$ follows from $p \wedge s$, $q \to \neg r$, and $\neg s \vee q$.**

| | | |
|---|---|---|
| 1. | $p \wedge s$ | Given |
| 2. | $q \to \neg r$ | Given |
| 3. | $\neg s \vee q$ | Given |
| | | |
| 17. | $s$ | ? |
| 18. | $\neg\neg s$ | Double Negation: 17 |
| 19. | $q$ | $\vee$ Elim: 3, 18 |
| 20. | $\neg r$ | MP: 2, 19 |

# Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

| | | |
|---|---|---|
| 1. | $p \wedge s$ | Given |
| 2. | $q \rightarrow \neg r$ | Given |
| 3. | $\neg s \vee q$ | Given |
| 17. | $s$ | $\wedge$ Elim: 1 |
| 18. | $\neg\neg s$ | Double Negation: 17 |
| 19. | $q$ | $\vee$ Elim: 3, 18 |
| 20. | $\neg r$ | MP: 2, 19 |

No holes left! We just need to clean up a bit.

# Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

| | | |
|---|---|---|
| 1. | $p \wedge s$ | Given |
| 2. | $q \rightarrow \neg r$ | Given |
| 3. | $\neg s \vee q$ | Given |
| 4. | $s$ | $\wedge$ Elim: 1 |
| 5. | $\neg\neg s$ | Double Negation: 4 |
| 6. | $q$ | $\vee$ Elim: 3, 5 |
| 7. | $\neg r$ | MP: 2, 6 |

# Important: Applications of Inference Rules

- You can use equivalences to make substitutions

  of any sub-formula.

  e.g. $(p \rightarrow r) \vee q \equiv (\neg p \vee r) \vee q$

- Inference rules only can be applied to whole formulas (not correct otherwise).

  e.g. 1. $p \rightarrow r$ given

  2. ~~$(p \vee q) \rightarrow r$ intro $\vee$ from 1.~~

  **Does not follow! e.g . p=F, q=T, r=F**

# Recall: Propositional Inference Rules

Two inference rules per binary connective, one to eliminate it and one to introduce it

**Elim ∧**

$$\frac{A \wedge B}{\therefore A, B}$$

**Intro ∧**

$$\frac{A \,;\, B}{\therefore A \wedge B}$$

**Elim ∨**

$$\frac{A \vee B \,;\, \neg A}{\therefore B}$$

**Intro ∨**

$$\frac{A}{\therefore A \vee B,\ B \vee A}$$

**Modus Ponens**

$$\frac{A \,;\, A \rightarrow B}{\therefore B}$$

**Direct Proof**

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

# Recall: New Perspective

Rather than comparing A and B as columns,

zooming in on just the rows where A is true:

| ***p*** | ***q*** | **A** | **B** |
|---|---|---|---|
| T | T | T | T |
| T | F | T | T |
| F | T | F | |
| F | F | F | |

Given that A is true, we see that B is also true.

A ® B

# Recall: New Perspective

Rather than comparing A and B as columns,

zooming in on just the rows where B is true:

| $p$ | $q$ | **A** | **B** | **A → B** |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | T | T | T |
| F | T | F | T | T |
| F | F | F | F | T |

When we zoom out, what have we proven?

$$(A \rightarrow B) \equiv \mathbf{T}$$

# Recall: Propositional Inference Rules

Two inference rules per binary connective, one to eliminate it and one to introduce it

Elim ∧

$$\frac{A \wedge B}{\therefore A, B}$$

Intro ∧

$$\frac{A \,;\, B}{\therefore A \wedge B}$$

Elim ∨

$$\frac{A \vee B \,;\, \neg A}{\therefore B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

Modus Ponens

$$\frac{A \,;\, A \rightarrow B}{\therefore B}$$

Direct Proof

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

Not like other rules

# To Prove An Implication: $A \rightarrow B$

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

- We use the direct proof rule
- The "pre-requisite" $A \Rightarrow B$ for the direct proof rule is a proof that "Assuming A, we can prove B."
- The direct proof rule:
  If you have such a proof, then you can conclude that $A \rightarrow B$ is true

# Proofs using the direct proof rule

Show that $p \to r$ follows from $q$ and $(p \land q) \to r$

1. $q$ Given
2. $(p \land q) \to r$ Given

This is a proof of $p \to r$

3.1. $p$ Assumption
3.2.
**3.3.** $r$ **??**

If we know $p$ is true... Then, we've shown $r$ is true

3. $p \to r$ Direct Proof

# Proofs using the direct proof rule

Show that $p \rightarrow r$ follows from $q$ and $(p \wedge q) \rightarrow r$

1. $\boldsymbol{q}$ Given
2. $(\boldsymbol{p} \wedge \boldsymbol{q}) \rightarrow \boldsymbol{r}$ Given
   - 3.1. $\boldsymbol{p}$ Assumption
   - 3.2. $\boldsymbol{p} \wedge \boldsymbol{q}$ Intro $\wedge$: **1, 3.1**
   - **3.3.** $\boldsymbol{r}$ **MP: 2, 3.2**
3. $\boldsymbol{p} \rightarrow \boldsymbol{r}$ Direct Proof

# Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

**There MUST be an application of the Direct Proof Rule (or an equivalence) to prove this implication.**

**Where do we start? We have no givens…**

# Example

Prove: $(p \land q) \rightarrow (p \lor q)$

| | | |
|---|---|---|
| 1.1. | $p \land q$ | Assumption |
| 1.9. | $p \lor q$ | ?? |
| 1. | $(p \land q) \rightarrow (p \lor q)$ | Direct Proof |

# Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

| | | |
|---|---|---|
| 1.1. | $p \wedge q$ | Assumption |
| 1.2. | $p$ | Elim $\wedge$: 1.1 |
| 1.3. | $p \vee q$ | Intro $\vee$: 1.2 |
| 1. | $(p \wedge q) \rightarrow (p \vee q)$ | Direct Proof |

# One General Proof Strategy

1. Look at the rules for introducing connectives to see how you would build up the formula you want to prove from pieces of what is given

2. Use the rules for eliminating connectives to break down the given formulas so that you get the pieces you need to do 1.

3. Write the proof beginning with what you figured out for 2 followed by 1.

# Example

Prove: $((p \to q) \land (q \to r)) \to (p \to r)$

# Example

Prove: $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

**1.1.** $(\boldsymbol{p} \rightarrow \boldsymbol{q}) \wedge (\boldsymbol{q} \rightarrow \boldsymbol{r})$ **Assumption**

**1.?** $\boldsymbol{p} \rightarrow \boldsymbol{r}$

**1.** $\big((\boldsymbol{p} \rightarrow \boldsymbol{q}) \wedge (\boldsymbol{q} \rightarrow \boldsymbol{r})\big) \rightarrow (\boldsymbol{p} \rightarrow \boldsymbol{r})$ **Direct Proof**

# Example

Prove: $((p \to q) \land (q \to r)) \to (p \to r)$

1.1. $(p \to q) \land (q \to r)$ **Assumption**

1.2. $p \to q$ **∧ Elim: 1.1**

1.3. $q \to r$ **∧ Elim: 1.1**

1.? $p \to r$

1. $\big((p \to q) \land (q \to r)\big) \to (p \to r)$ **Direct Proof**

# Example

Prove: $((p \to q) \land (q \to r)) \to (p \to r)$

1.1. $(p \to q) \land (q \to r)$ **Assumption**

1.2. $p \to q$ **∧ Elim: 1.1**

1.3. $q \to r$ **∧ Elim: 1.1**

1.4.1. $p$ **Assumption**

1.4.? $r$

1.4. $p \to r$ **Direct Proof**

1. $((p \to q) \land (q \to r)) \to (p \to r)$ **Direct Proof**

# Example

Prove: $((p \to q) \land (q \to r)) \to (p \to r)$

| | | |
|---|---|---|
| 1.1. | $(p \to q) \land (q \to r)$ | Assumption |
| 1.2. | $p \to q$ | $\land$ Elim: 1.1 |
| 1.3. | $q \to r$ | $\land$ Elim: 1.1 |
| 1.4.1. | $p$ | Assumption |
| 1.4.2. | $q$ | MP: 1.2, 1.4.1 |
| 1.4.3. | $r$ | MP: 1.3, 1.4.2 |
| 1.4. | $p \to r$ | Direct Proof |
| 1. | $((p \to q) \land (q \to r)) \to (p \to r)$ | Direct Proof |

# Minimal Rules for Propositional Logic

Can get away with just these:

Elim ∧: $$\frac{A \wedge B}{\therefore A,\ B}$$

Intro ∧: $$\frac{A\ ;\ B}{\therefore A \wedge B}$$

Elim ∨: $$\frac{A \vee B\ ;\ \neg A}{\therefore B}$$

Intro ∨: $$\frac{A}{\therefore A \vee B,\ B \vee A}$$

Modus Ponens: $$\frac{A\ ;\ A \rightarrow B}{\therefore B}$$

Direct Proof: $$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

Excluded Middle: $$\frac{}{\therefore A \vee \neg A}$$

# More Rules for Propositional Logic

More rules makes proofs easier

| Rule | Premises | Conclusion |
|---|---|---|
| Elim ∧ | $A \wedge B$ | $\therefore A, B$ |
| Intro ∧ | $A \,;\, B$ | $\therefore A \wedge B$ |
| Elim ∨ | $A \vee B \,;\, \neg A$ | $\therefore B$ |
| Intro ∨ | $A$ | $\therefore A \vee B, B \vee A$ |
| Modus Ponens | $A \,;\, A \rightarrow B$ | $\therefore B$ |
| Direct Proof | $A \Rightarrow B$ | $\therefore A \rightarrow B$ |
| Excluded Middle | | $\therefore A \vee \neg A$ |
| Equivalent | $A \equiv B \,;\, B$ | $\therefore A$ |

# Example Formal Proof:

Show that $\neg p$ follows from $\neg(\neg r \vee t), \neg q \vee \neg s,$ and $(p \rightarrow q) \wedge (r \rightarrow s)$

| | |
|---|---|
| 1. $\neg(\neg r \vee t)$ | Given |
| 2. $\neg q \vee \neg s$ | Given |
| 3. $(p \rightarrow q) \wedge (r \rightarrow s)$ | Given |
| 4. $r \wedge t$ | Equivalent 1. (double negation and demorgans law) |
| 5. $r \rightarrow s$ | Elim And 3. |
| 6. $r$ | Elim And 4. |
| 7. $s$ | MP, 5, 6. |
| 8. $\neg\neg s$ | Equivalent 7 (double negation). |
| 9. $\neg q$ | Elim or, 2, 8. |
| 10. $p \rightarrow q$ | Elim And, 3. |
| 11. $\neg q \rightarrow \neg p$ | Contrapositive, 10 |
| 12. $\neg p$ | MP, 9, 11. |