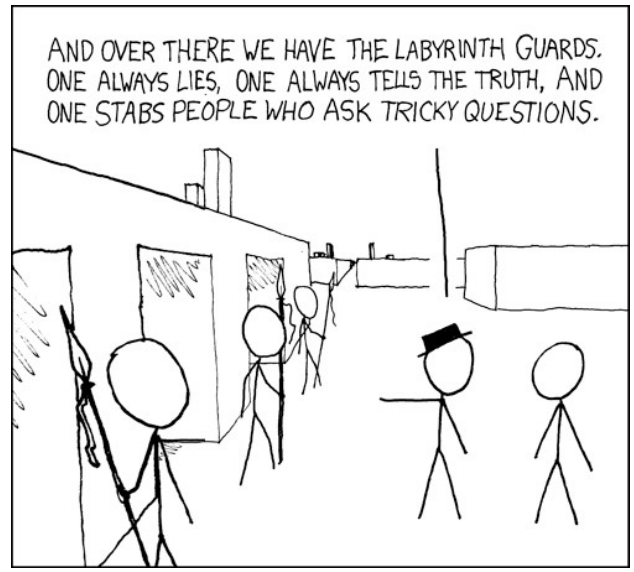

Propositional Logic, Equivalences

CSE 311: Foundations of Computing I
Lecture 2

Announcements

- HW1 is posted on the course website
  - Due Friday, 11:59 pm
  - Submit on Gradescope
- OH begin on Monday

Review

Recall: Atomic Propositions

- Atomic Propositions are true or false statements that cannot be broken down any further
- Propositional variables: p, q, r, s …

Recall: Logical Connectives

Name	Logical Symbol
Not	$\neg p$
And	$p \wedge q$
Or	$p \vee q$
XOR	$p \oplus q$
Implication	$p \rightarrow q$
Biconditional	$p \leftrightarrow q$

Recall: Implication

"If it's raining, then I have my umbrella"

p: It is raining q: I have my umbrella

$p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Equivalently:

- Whenever it is raining, I have my umbrella.
- It is raining only if I have my umbrella.
- For it to be raining, it is necessary that I have my umbrella.

Recall: Compound Proposition

- Unless I go to a café or to campus, I do not drink coffee, but also I don't go to cafés.

- What does this mean? Find the atomic propositions and translate to logic.
- p: I go to a café
- q: I go to campus
- r: I drink coffee

$$(\neg(p \vee q) \rightarrow \neg r) \wedge \neg p$$

Recall: Truth Table (from section)

p	q	r	$p \vee q$	$\neg(p \vee q)$	$\neg r$	$\neg(p \vee q) \to \neg r$	$\neg p$	$(\neg(p \vee q) \to \neg r) \wedge (\neg p)$
T	T	T	T	F	F	T	F	F
T	T	F	T	F	T	T	F	F
T	F	T	T	F	F	T	F	F
T	F	F	T	F	T	T	F	F
F	T	T	T	F	F	T	T	T
F	T	F	T	F	T	T	T	T
F	F	T	F	T	F	F	T	F
F	F	F	F	T	T	T	T	T

Normal Forms

Normal Forms

- Given any truth table, can we create a propositional logic expression that generates that truth table?

p	q	$F(p,q)$
T	T	T
T	F	F
F	T	T
F	F	F

Disjunctive Normal Form (DNF)

- ORs of ANDs

1. Read the true rows of the truth table
2. AND together all settings in a true row
3. OR together the true rows

p	q	$F(p,q)$
T	T	T
T	F	F
F	T	T
F	F	F

$$F(p,q) \equiv (p \wedge q) \vee (\neg p \wedge q)$$

Conjunctive Normal Form (CNF)

- ANDs of ORs

1. Read the false rows of the truth table
2. OR together **the negation** of all settings in a false row
3. AND together the false rows

p	q	$F(p,q)$
T	T	T
T	F	F
F	T	T
F	F	F

$$F(p,q) \equiv (\neg p \lor q) \land (p \lor q)$$

Normal Forms

- Don’t simplify CNF / DNF further
- These are standard forms – everyone's CNF / DNF formulas will be the same (up to commutativity)

Normal Form Example

Write the CNF and DNF of the following truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

DNF: $(p \land q) \lor (\neg p \land q) \lor (\neg p \land \neg q)$

CNF: $(\neg p \lor q)$

Logical Equivalence

Logical Equivalence

- Definition:

Two propositions are logically equivalent if they have identical truth values.

- The notation for A and B being logically equivalent is $\mathrm{A} \equiv \mathrm{B}$.

- Examples:
- $p \vee q \equiv q \vee p$
- $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$=$ vs. $\equiv$ vs. $\leftrightarrow$

- $\mathrm{A} = \mathrm{B}$ means A, B are the exact same "strings" or "references".
- E.g. $p \vee q = p \vee q$, but $p \vee q \neq q \vee p$. We hardly use $=$ with propositions!
- $\mathrm{A} \equiv \mathrm{B}$ is an assertion that A, B always have the same truth value.
- E.g. $p \vee q \equiv q \vee p$.
- $\mathrm{A} \leftrightarrow \mathrm{B}$ is a proposition that might be true or false.
- E.g. $p \vee q \leftrightarrow p \wedge q$ is a false proposition.
- $A \equiv B$ has the same meaning as $(A \leftrightarrow B) \equiv T$.

Tautologies:

Terminology: A compound proposition is a…

- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

$p \vee \neg p$

This is a tautology. It's called the "law of the excluded middle".
If p is true, then $p \vee \neg p$ is true. If p is false, then $p \vee \neg p$ is true.

$p \oplus p$

This is a contradiction. It's always false no matter what truth value p takes on.

$(p \rightarrow r) \wedge p$

This is a contingency. When p=T, r=T, $(T \rightarrow T) \wedge T$ is true.
When p=T, r=F, $(T \rightarrow F) \wedge T$ is false.

Proving Logical Equivalence

Motivation

- Given two propositions, we would like to know if they are equivalent.
- E.g. one developer wrote $\text{if}\left((p\ \&\&\ p)\ ||\ p\right)\{\dots\}$.

Another developer wrote $\text{if}\,(p)\,\{\dots\}$.

You want to confirm if those are the same.

- Given a complicated proposition, we would like to find a simpler proposition that it's equivalent to.

Strategy 1: Truth Tables

- Make a truth table for the two propositions and check if they are the same.
- p vs. $(p \wedge p) \vee p$

p	$p \wedge p$	$(p \wedge p) \vee p$
T	T	T
F	F	F

Strategy 1: Truth Tables

- Truth tables do let us check if two propositions are equivalent.
- Truth tables don't give us a good way to start from a complicated proposition, and simplify it.
- What would be the runtime of this algorithm?

Strategy 2: Manipulating Expressions

- Instead, we are going to learn logical equivalence rules to help us simplify expressions.

- Similar to algebra, where we can apply rules to transform expression:
- $(x+2)(x+3) = x^2 + 2x + 3x + 6$ Distributivity
- $= x^2 + 5x + 6$ Adding like terms

- For each rule, we will understand why it's true, and practice using it.

Logical Equivalence Rules

All the rules:

For every propositions p, q, r the following hold:

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$
- **DeMorgan's Laws**
  - $\neg(p \vee q) \equiv \neg p \wedge \neg q$
  - $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- **Double Negation**

  $\neg\neg p \equiv p$
- **Law of Implication**

  $p \to q \equiv \neg p \vee q$
- **Contrapositive**

  $p \to q \equiv \neg q \to \neg p$

Double Navigation

-

$$\neg\neg p \equiv p$$

- I am not not loving propositional logic.
- I am loving propositional logic.

De Morgan's Laws: Intuition

- Consider the following sentences:
- I don't like apples or mangoes. $\neg(p \vee q)$
- I don't like apples, and I don't like mangoes. $\neg p \wedge \neg q$

- Are they logically equivalent? Intuitively, yes

De Morgan's Laws

-
-

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

De Morgan's Laws

- Example: $\neg(p \vee q) \equiv \neg p \wedge \neg q$

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

De Morgan’s Laws

- `if (!(front != null && value > front.data)) {…}`

- `if (front == null || value <= front.data) {…}`

Law of Implication

- Implications are unusual. Can we write them using ANDs ORs & NOTs?

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Law of Implication: Intuition

- $p \rightarrow q \equiv \neg p \vee q$

- If it is raining, then I have my umbrella.
- It is not raining, or I have my umbrella.

Law of Implication

-

$$p \to q \equiv \neg p \lor q$$

p	q	$\neg p$	$p \to q$	$\neg p \lor q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Converse & Contrapositive

- Implication: $p \rightarrow q$ If it's raining, I have my umbrella.
- Converse: $q \rightarrow p$ If I have my umbrella, it's raining.
- Contrapositive: $\neg q \rightarrow \neg p$ If I don't have my umbrella, it's not raining.

Converse & Contrapositive

- Implication: $p \rightarrow q$ Converse: $q \rightarrow p$ Contrapositive: $\neg q \rightarrow \neg p$
- How do these relate?

p	q	$p \rightarrow q$	$q \rightarrow p$	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$
T	T	T	T	F	F	T
T	F	F	T	F	T	F
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Contrapositive

-

$$p \to q \equiv \neg q \to \neg p$$

Contrapositive: Intuition

- $p \to q \equiv \neg q \to \neg p$

- If an animal is a cat, then it is a mammal.
- If an animal is not a mammal, then it’s not a cat.

Commutativity

- $$p \vee q \equiv q \vee p$$
- $$p \wedge q \equiv q \wedge p$$

- It is raining or it is June.
- It is June or it is raining.

Associativity

-
-

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

$$p \wedge (q \wedge r) \equiv p \wedge (q \wedge r)$$

- They perform at 3:00 and 5:00, and also 8:00.
- They perform at 3:00, and also 5:00 and 8:00.

WARNING

Only apply associativity when all connectives are AND, or all connectives are OR

Exercise

- Prove that $p \rightarrow q \equiv \neg q \rightarrow \neg p$ using the logical equivalences rules we've discussed so far.

  Do not use contrapositive in the proof.

- $\neg q \rightarrow \neg p \equiv \neg\neg q \vee \neg p$ Law of Implication
- $\equiv q \vee \neg p$ Double Negation
- $\equiv \neg p \vee q$ Commutativity
- $\equiv p \rightarrow q$ Law of Implication

Distributivity: Intuition

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

- You go to class, and you read the notes or the textbook.
- You go to class and read the notes, or you go to class and you read the textbook.

Distributivity

- $$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
- $$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Identity

-
-

$$p \wedge \mathrm{T} \equiv p$$

$$p \vee \mathrm{F} \equiv p$$

p	$p \wedge \mathrm{T}$	$p \vee \mathrm{F}$
T	T	T
F	F	F

Domination

-
-

$$p \vee \mathrm{T} \equiv \mathrm{T}$$

$$p \wedge \mathrm{F} \equiv \mathrm{F}$$

p	$p \vee \mathrm{T}$	$p \wedge \mathrm{F}$
T	T	F
F	T	F

Idempotency

- $p \vee p \equiv p$
- $p \wedge p \equiv p$

p	$p \vee p$	$p \wedge p$
T	T	T
F	F	F

Negation Intuition

- $p \vee \neg p \equiv \mathrm{T}$
- $p \wedge \neg p \equiv \mathrm{F}$

- It is raining or it is not raining. Always true
- It is raining and it is not raining. Always false

Negation

-
-

$$p \vee \neg p \equiv \mathrm{T}$$

$$p \wedge \neg p \equiv \mathrm{F}$$

p	$p \vee \neg p$	$p \wedge \neg p$
T	T	F
F	T	F

Absorption

- $p \lor (p \land q) \equiv p$
- $p \land (p \lor q) \equiv p$

- Exercise: Build the truth tables to confirm.

Absorption

p	q	$p \wedge q$	$p \vee (p \wedge q)$	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	F	T	F
F	F	F	F	F	F

Logical Equivalence Examples

Ex 1: Prove $(p \land q) \lor (\neg p \land q) \lor (\neg p \land \neg q) \equiv p \rightarrow q$

$(p \land q) \lor (\neg p \land q) \lor (\neg p \land \neg q) \equiv \big((p \lor \neg p) \land q\big) \lor (\neg p \land \neg q)$	Distributivity
$\equiv (\mathrm{T} \land q) \lor (\neg p \land \neg q)$	Negation
$\equiv q \lor (\neg p \land \neg q)$	Identity
$\equiv (q \lor \neg p) \land (q \lor \neg q)$	Distributivity
$\equiv (q \lor \neg p) \land \mathrm{T}$	Negation
$\equiv q \lor \neg p$	Identity
$\equiv \neg p \lor q$	Commutativity
$\equiv p \rightarrow q$	L.O.I

Caveat 1: Associativity & Commutativity

- Show that $(p \lor q) \lor (r \lor s) \equiv r \lor (q \lor s) \lor p$, following rules exactly.

$(p \lor q) \lor (r \lor s) \equiv p \lor \big(q \lor (r \lor s)\big)$	Associativity
$\equiv p \lor \big(q \lor (s \lor r)\big)$	Commutativity
$\equiv p \lor \big((q \lor s) \lor r\big)$	Associativity
$\equiv \big((q \lor s) \lor r\big) \lor p$	Commutativity
$\equiv \big(r \lor (q \lor s)\big) \lor p$	Commutativity
$\equiv r \lor (q \lor s) \lor p$	Order of Operations

Caveat 1: Associativity & Commutativity

- Show that $(p \lor q) \lor (r \lor s) \equiv r \lor (q \lor s) \lor p$.
- We will allow abbreviated associativity & commutativity steps.

- $(p \lor q) \lor (r \lor s) \equiv r \lor (q \lor s) \lor p$ Associativity & Commutativity

Caveat 1: Associativity & Commutativity

- Show that $\neg p \vee p \equiv \mathbf{T}$.

Showing all steps:

$\neg p \vee p \equiv p \vee \neg p$ Commutativity

$\equiv \mathbf{T}$ Negation

What we allow:

$\neg p \vee p \equiv \mathbf{T}$ Negation

Caveat 2: Applying a rule twice

- Expand $(p \rightarrow q) \vee (q \rightarrow r)$ using the Law of Implication.

- Showing all steps:

$(p \rightarrow q) \vee (q \rightarrow r) \equiv (\neg p \vee q) \vee (q \rightarrow r)$	Law of Implication
$\equiv (\neg p \vee q) \vee (\neg q \vee r)$	Law of Implication

- What we allow:

$(p \rightarrow q) \vee (q \rightarrow r) \equiv (\neg p \vee q) \vee (\neg q \vee r)$	Law of Implication (x2)

Caveat 3: Applying rules to any proposition

- We can apply equivalence rules to any proposition.
- $(p \vee q \wedge r) \vee (p \vee q \wedge r) \equiv p \vee q \wedge r$ Idempotency
- $\neg\neg(r \rightarrow \neg q) \equiv r \rightarrow \neg q$ Double Negation
- $\neg\big((p \vee q) \wedge s\big) \equiv \neg(p \vee q) \vee \neg s$ DeMorgan's Law