CSE 311 : Midterm Exam

1. Translation

Let your domain of discourse be positive integers.

For this problem, you may use the predicates

- $\texttt{Even}(x)$ which is true if and only if x is even.
- $\texttt{Odd}(x)$ which is true if and only if x is odd.
- $\texttt{PrimePower}(x)$ which is true if and only if x is "a prime power" (which means the prime factorization of x is p^a for a prime number p and an integer a. $125 = 5^3$ is a prime power, $24 = 2^3 \cdot 3$ is not a prime power)
- $\texttt{PowerOfTwo}(x)$, which is true if and only if x is a power of 2 (i.e., $x = 2^c$ for some integer c)
- standard math predicates (e.g. $=, \neq, <, >, \geq, \ldots$)

(a) Translate the following predicate logic statement into English. Your translation must be natural.

$\forall x(\texttt{PrimePower}(x) \to \texttt{Odd}(x) \vee \texttt{PowerOfTwo}(x))$

(b) Translate "There is more than one prime power" into predicate logic.

(c) Find an equivalent statement to the one below by taking the contrapositive of the implication inside. Give your final answer in English; you do not need to show work.

"For every integer, if it is even then it is a power of two or not a prime power."

(d) Negate the following predicate logic sentence. In your final answer, negations should only be applied to single predicates.

$\exists x \forall y([\texttt{Even}(y) \vee \texttt{PrimePower}(y)] \to [\texttt{Even}(x) \wedge \texttt{Odd}(y)])$

2. Even Circuits Are Fun

The function multiple-of-three takes in two inputs: $(x_1x_0)_2$ and outputs 1 iff $3 \mid (x_1x_0)_2$. $(x_1x_0)_2$ represents the binary number where the first bit is x_1 and the second bit is x_2

(a) (5 points) Draw a table of values (e.g. a truth table) for multiple-of-three.

(b) (5 points) Write multiple-of-three as a sum-of-products. (DNF)

(c) (5 points) Write multiple-of-three as a product-of-sums. (CNF)

3. Number Theory Proof

For the questions below you may use the definitions of modular equivalence and divides and algebra.

You may not use results from the number theory formula sheet or theorems proven in class (though you may emulate those proofs!)

Finally, you may also use this fact without proving it:

Fact 1: For any two integers x, y and any prime p: if $xy \equiv 0 \pmod{p}$ then $p|x$ or $p|y$.

(a) Disprove this statement with a counterexample.

For every integer n, $ab \equiv 0 \pmod{n}$ implies $a \equiv 0 \pmod{n}$ or $b \equiv 0 \pmod{n}$. (Hint: you will need to choose n to be a composite number).

(b) Prove that for every prime p: If $ab \equiv 0 \pmod{p}$, then $a \equiv 311p \pmod{p}$ or $b \equiv 0 \pmod{p}$.

4. Induction Proof

Prove that $6 | \left(10^{2n} + 2\right)$ for all $n \in \mathbb{Z}^+$ using induction on n.

Recall that $\mathbb{Z}^+$ is the positive integers (i.e., starting at 1).

Don't forget to define your predicate as part of your proof!

Hint: $198 = 6 \cdot 33$.

5. First Proof [12 points]

(a) Prove $(A \cup B) \setminus (A \cap B) \subseteq [A \setminus (A \cap B)] \cup [B \setminus (A \cap B)]$
You must format your proof as an English proof and structure your proof by introducing arbitrary element(s) of sets as appropriate.
We recommend drawing a picture of the sets for yourself so you see why the statement is true. [7 points]

(b) From (only) what you've written above, can you conclude: $(A \cup B) \setminus (A \cap B) = [A \setminus (A \cap B)] \cup [B \setminus (A \cap B)]$ (we've changed the subset from (a) to an equals sign).

If you can conclude the new statement, briefly (1-2 sentences) explain why.
If you cannot conclude the new statement, what statement do you still need to prove to get the new conclusion? (you can give you answer in English, set notation, or predicate logic notation – whichever you find most convenient). [2 points]

(c) Disprove the following statement: $(A \cup B) \setminus (A \cap B) = [(A \cup B) \setminus A] \cap [(A \cup B) \setminus B]$ [3 points