Homework 3: English Proofs

Due date: Friday, July 12th at 11:59 PM
If you work with others (and you should!), remember to follow the collaboration policy outlined in the syllabus.
In general, you are graded on your work's clarity and accuracy. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.
We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we expect. You can have longer explanations, but explanations significantly longer than necessary may receive deductions.

1. Mary, Mary, Quite Contrary [10 points]

When direct proofs fail, our logical equivalences can come to the rescue. Consider the statement

For every integer k, if $k^3 + 9$ is odd then k is even

Proving this directly is not easy (try it for yourself to see!). Instead, we will prove the contrapositive of this statement.

(a) Write the contrapositive of the given statement (in English). [2 points]

(b) Write a proof by contrapositive (do an English proof) of the given statement. [8 points]

2. Divides Dungeon [8 points]

Write an English proof to show that if 10 divides $(x + 3)$ (i.e. $10|(x + 3)$) for an integer x, then x is odd. Recall that English proofs don't have domains of discourse, so you need to state the types for your variables when you introduce them.

3. Greatest Common Dilemma [16 points]

Bezout's Theorem (one of the theorems in the optional number theory content) tells us that if a and b are positive integers, then there exist some integers s and t such that gcd$(a, b) = sa + tb$.

However, the converse isn't always true: there could exist some integers s and t such that $d = sa + tb$, but d isn't necessarily gcd(a, b). In this problem, we will see a special case where the converse does hold. You may use without proof that if any integer k satisfies $k|1$, then k must be either 1 or -1.

(a) For all **positive** integers a and b, prove the following claim: if there exist some integers s and t such that $sa + tb = 1$, then gcd$(a, b) = 1$. [12 points]

Hint: The facts about GCD that you will need for this problem are that if $a =$ gcd(b, c) then $a|b$ and $a|c$, and it is the largest integer that does this. This is also an exists claim, so you will want to have in your proof something like "suppose there exists some integers s and t such that $sa + tb = 1$"

(b) Use part (a) to show that gcd$(n, n + 1) = 1$ for all positive integers n. [4 points]

4. Modulo Mystery

Let the domain of discourse be the integers. Write an English proof of the following claim: for any integers a and b; if a is congruent to 7 modulo 12 and b is congruent to 5 modulo 9, then $a - b$ is congruent to 5 modulo 3.

5. Prove or Disprove [12 points]

Let a, b, c be arbitrary integers such that $a|b$ and $b|c$. For the following questions, if the statement is true, write a proof. If it is false, disprove it (you will provide a counterexample in that case).

(a) Is it true that $ab|c$?

(b) Is it true that $a|c$?

6. Euclid's Extended Adventure [14 points]

(a) Compute the multiplicative inverse of $19 \pmod{165}$. Use the Extended Euclidean algorithm, showing the tableau and the sequence of substitutions.

Express your final answer as an integer between 1 and 165 inclusive. [6 points]

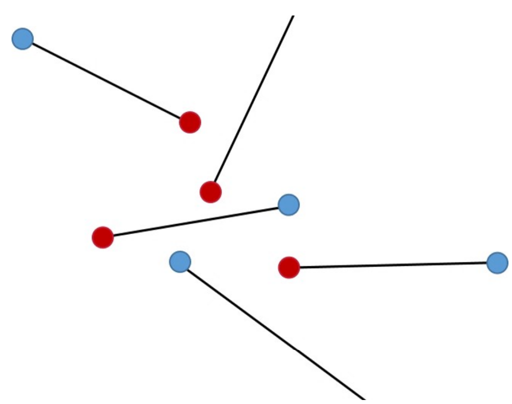
(b) Find **all** integer solutions to

$$19x \equiv 31 \pmod{165}$$

Your final answer must be in set-builder notation, and cannot use divides or the modular equivalence symbol. (for example $\{x : x = k \cdot 121 + 13 \text{ for some } k \in \mathbb{Z}\}$). Show your work for solving the equation, but you do not need to repeat any work from part (a). [8 points]

7. Extra Credit: Match Me if You Can

In this problem, you will show that given n red points and n blue points in the plane such that no three points lie on a common line, it is possible to draw line segments between red-blue pairs so that all the pairs are matched and none of the line segments intersect. Assume that there are n red and n blue points fixed in the plane.



A *matching* M is a collection of n line segments connecting distinct red-blue pairs. The *total length* of a matching M is the sum of the lengths of the line segments in M. Say that a matching M is *minimal* if there is no matching with a smaller total length.

Let $\mathsf{IsMinimal}(M)$ be the predicate that is true precisely when M is a minimal matching. Let $\mathsf{HasCrossing}(M)$ be the predicate that is true precisely when there are two line segments in M that cross each other.

Give an argument in English explaining why there must be at least one matching M so that $\mathsf{IsMinimal}(M)$ is true, i.e.

$$\exists M \mathsf{IsMinimal}(M))$$

Give an argument in English explaining why

$$\forall M(\mathsf{HasCrossing}(M) \to \neg\mathsf{IsMinimal}(M))$$

Then, use the two results above to give a proof of the statement:

$$\exists M \neg \mathsf{HasCrossing}(M).$$

8. Feedback [1 point]

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment (excluding any extra credit questions, if applicable)? Report your estimate to the nearest hour.
- Which problem did you spend the most time on?
- Any other feedback for us?