Homework 5: Number Theory, Induction

Due date: Friday, July 19th at 11:59 PM
If you work with others (and you should!), remember to follow the collaboration policy outlined in the syllabus.
In general, you are graded on your work's clarity and accuracy. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.
We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we expect. You can have longer explanations, but explanations significantly longer than necessary may receive deductions.

For this homework, all proofs must be done by one of the types of induction

1. Getting Stronger with Induction [20 points]

Suppose we have the following recursively defined function, $T(n)$:

$$T(n) = \begin{cases} 1 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ (9n^2 - 9n) \cdot T(n-2) & \text{if } n \text{ is a natural number and } n \geq 2 \end{cases}$$

Use induction to prove that for all integers n with $n \geq 0$, $T(n) = 3^n n!$.
Recall that for a positive integer n, $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$, and that $0! = 1$.

2. The Leaves Don't Fall Far From The... `Tree` [20 points]

In CSE123, you saw a recursive definition of trees. That definition looks a little different from what we saw in class.

The following definition is analogous to what you saw in 123. For this problem, we will consider binary trees defined as follows:

Basis Step: $(\texttt{null}, \bullet, \texttt{null})$ is a `tree`.
Recursive Step: If L, R are `trees` then $(L, \bullet, R)$ is also a `tree`

We will also use the following recursively defined functions for leaves:

$$\texttt{leaves}((\texttt{null}, \bullet, \texttt{null})) = 1$$
$$\texttt{leaves}((L, \bullet, R)) = \texttt{leaves}(L) + \texttt{leaves}(R) \qquad \text{for two arbitrary trees } L, R$$

And height:

$$\texttt{height}((\texttt{null}, \bullet, \texttt{null})) = 0$$
$$\texttt{height}((L, \bullet, R)) = 1 + \max(\texttt{height}(L), \texttt{height}(R)) \qquad \text{for two arbitrary trees } L, R$$

Show that for all `trees` t: $\texttt{leaves}(t) \leq 2^{\texttt{height}(t)}$

3. Pro-duction [20 points]

Prove that $(1 - 1/4)(1 - 1/9) \cdots (1 - 1/n^2) = \frac{n+1}{2n}$ for all integers $n \geq 2$.

4. Well that's odd... [20 points]

Prove that for every positive integer n you can find an integer a and odd number b satisfying $n = 2^a b$.

5. Feedback [1 point]

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment (excluding any extra credit questions, if applicable)? Report your estimate to the nearest hour.
- Which problem did you spend the most time on?
- Any other feedback for us?