Homework 2: Circuits, Boolean Algebra, and More Logic

Due date: Friday, July 5th at 11:59 PM
If you work with others (and you should!), remember to follow the collaboration policy outlined in the syllabus.
In general, you are graded on your work's clarity and accuracy. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.
We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we expect. You can have longer explanations, but explanations significantly longer than necessary may receive deductions.

1. The Many Citations of Mathematicians [16 points]

Let the domain of discourse be mathematicians. We define the predicate $\mathsf{Cited}(x, y)$ to mean that x cited y. We define the predicate $\mathsf{FormallyTrained}(x)$ to mean that x was formally trained in mathematics. We define the predicate $\mathsf{DiscoveredCalculus}(x)$ to mean that x discovered calculus. You can also assume an "$=$" operator that is true when x and y are the same mathematician, and an "$\neq$" operator that is true when x and y are different mathematicians.

Translate each of the following English statements into predicate logic. Do not simplify.

(a) There is a mathematician whom every mathematician has cited. [4 Points]

(b) All formally trained mathematicians have cited a mathematician who was not formally trained. [4 Points]

(c) There is a mathematician who has cited themselves, but has not cited any other mathematician. [4 Points]

(d) There is exactly one mathematician who discovered calculus. [4 Points]

2. Odd numbers, Even results [12 points]

(a) Let the domain of discourse be integers. Define the predicates $\text{Odd}(x) := \exists k(x = 2k + 1)$, and $\text{Even}(x) := \exists k(x = 2k)$. [4 points]

Translate the following claim to predicate logic:

For all odd integers n and m, $n + 3m$ is even.

(b) Prove that the claim is true. [8 points] For this problem, write an English proof

3. The Theory of Predicates [12 points]

Let the domain of discourse be students and courses in the computer science department. Define the predicates $\mathsf{Student}(x)$ to mean that x is a student and $\mathsf{Course}(y)$ to mean that y is a course. Define the predicate $\mathsf{Enjoys}(x, y)$ to mean that x enjoys y (where presumably x is a student and y is a course) and the predicate $\mathsf{TheoryBased}(y)$ to mean that y is based on theory (presumably a course).

Translate each of the following logical statements into English. You should not simplify. However, you should use the techniques shown in lecture for producing more natural translations when restricting domains and for avoiding the introduction of variable names when not necessary.

(a) $\exists y(\mathsf{Course}(y) \land \neg \mathsf{TheoryBased}(y))$ [3 Points]

(b) $\exists x(\mathsf{Student}(x) \land \forall y(\mathsf{Course}(y) \land \mathsf{TheoryBased}(y) \rightarrow \mathsf{Enjoys}(x, y)))$ [3 Points]

(c) $\forall y(\mathsf{Course}(y) \land \mathsf{TheoryBased}(y)) \rightarrow \exists x(\mathsf{Student}(x) \land \neg \mathsf{Enjoys}(x, y))$ [3 Points]

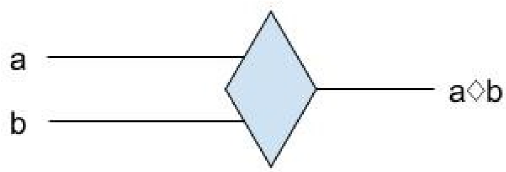
(d) $\forall x(\mathsf{Student}(x) \rightarrow \exists y(\mathsf{Course}(y) \land \mathsf{Enjoys}(x, y))$ [3 Points]

4. Shine Bright Like a Diamond [20 points]

Consider a new logical operator $\diamond$ (*diamond* in latex). For this problem, we define $a \diamond b$ by the following truth table:

A	B	$A \diamond B$
1	1	0
1	0	1
0	1	1
0	0	1

Even though this is a new operator, we can still use it to create circuits representing other operators we have seen before! Here is an example drawing of the diamond gate:



(a) Using only $\diamond$ gates and the input a, create a circuit whose output represents $\neg a$. You may use multiple copies of a as inputs if required.

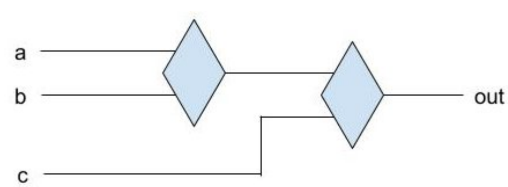
(b) Using only $\diamond$ gates and the inputs a and b, create a circuit that's output represents $a \land b$. You may use multiple copies of a, b as inputs if required.
Hint You might find it helpful for this problem to start by drawing a truth table with both $a \diamond b$ and $a \land b$.

(c) Using only $\diamond$ gates and the inputs a and b, create a circuit that's output represents $a \lor b$. You may use multiple copies of a, b as inputs if required.

4.1. More Diamonds!

Now that you have created these circuits, notice how the truth value of $a \diamond b$ is equivalent to $\neg(a \land b)$ (check the truth table to confirm this fact). This fact means we can perform translations from circuits and expressions using $\diamond$ to circuits and expressions using $\neg, \land, \lor$.

For example, consider this circuit:



This circuit can be read as $(a \diamond b) \diamond c$, which using the fact about $\diamond$ is equivalent to $\neg(\neg(a \land b) \land c)$.

(d) Translate your circuit for $a \land b$ from part (b) into propositional logic notation (i.e. notation using $\neg, \land$).
Do not simplify in this part; your final expression should only contain $\neg$ and $\land$ symbols and copies of a and b.

(e) As this expression represents $a \wedge b$, use a chain of logical equivalences to simplify your expression from the previous part into $a \wedge b$

5. What's wrong with this proof? [9 Points]

Consider the following statement:

For all real numbers x, y, and z, if $xy = yz$, then $x = z$.

And the following spoof (incorrect proof) of the statement:

> Let x and z be arbitrary real numbers and suppose that $x = z$. Let y be an arbitrary real number. Multiplying both sides of the equation by y, we obtain $xy = yz$.

(a) Why is the above proof incorrect?

Here, let's try again. This must be correct this time, right?

> Let x, y, z be arbitrary real numbers and suppose that $xy = yz$. Dividing both sides of the equation by y, we obtain $\frac{xy}{y} = \frac{yz}{y}$ which simplifies to $x = z$. Thus, our claim holds.

(b) Again, why is the above proof incorrect?

(c) Is the original statement true or false? If the statement is true, write a correct proof. If it is false, provide a counterexample.

6. Yay! Propositional logic proofs! [16 points]

Note that you may not use methods such as proof by contradiction, and may only use proof methods you have learned up to this point.

(a) Provide a formal proof for $((P \rightarrow Q) \rightarrow P) \rightarrow P$. Note that this is a *tautology* because it is true without any givens. This is known as Peirce's law, and holds philosophical significance in the theory of logic. [4 points]

(b) Provide a formal proof for $P \vee \neg P$ (known as the "law of the excluded middle"). This is also a tautology. You may not use the Law of the Excluded Middle. [4 points]

(c) Given $A \rightarrow (B \vee C)$ and $B \rightarrow \neg A$, provide a formal proof that $\neg A \vee C$. [4 points]

(d) Given $P \rightarrow \neg R$, $(Q \vee R) \wedge (R \vee S)$, and $Q \rightarrow ((R \vee S) \rightarrow V)$, it follows that $P \rightarrow V$. [4 points]

Note: Your proof for part d is only allowed to use the rules the Modus Ponens, Direct Proof, Intro $\wedge$, Elim $\wedge$, Intro $\vee$, Elim $\vee$. Note that Equivalent is not allowed this time. (*Hint*: Elim $\vee$ will be useful!)

7. Quack Detective [10 points]

Donald Duck has been accused of eating Micky's cookies last Sunday. Everybody knows the following rules are true:

- When Donald Duck eats cookies, he becomes very happy for the whole week.
- When Donald Duck is very happy, he smiles.

- If Donald Duck is left unattended in Micky's house and knows where Micky's cookies are, then he will eat them.

The following evidence is known about the events of last Sunday:

- Somebody ate Micky's cookies.
- Only Micky or Donald could have eaten Micky's cookies that day.
- Donald Duck has been seen smiling.
- Donald Duck was left unattended at Micky's house.

Minnie has made the following argument: "Donald Duck was seen smiling, which means he was happy, and everybody knows Donald Duck is happy when he eats Micky's cookies."

Micky has made the following argument: "Donald Duck was left unattended at my house, so he must have eaten them while I was gone."

(a) Fortunately for Donald, you've been appointed to defend him from these accusations. Explain why the above evidence and rules are not enough to soundly conclude that Donald is guilty. Identify two flaws in Minnie's chain of reasoning and then identify the flaw in Micky's reasoning. *Your answers should all relate to implications*. [4 points]

(b) You can have fun on this one. Find a plausible explanation for why Donald may be innocent despite the evidence. Your answer should provide a (potentially comedic) reason why each piece of evidence happened. [2 point]

(c) At the last minute, a key eyewitness, Goofie, testifies that he saw Donald frowning one evening this week. Make a logically sound argument using this fact, the evidence, and the rules, which proves Donald Duck is innocent. Furthermore, argue that Micky logically must have eaten the cookies. [4 points]

8. [Extra Credit] 1111+1111 = Integer Overflow?

In this question you will construct a binary calculator equipped with the addition operator. For ease of understanding and writing, feel free to use Java syntax for loop structures and method structures.
Assume you are given two binary integers in the form of `boolean[]`s, where the first one is some `a = boolean[n]`, of length n, the second is some `b = boolean[m]`, of length m. You may assume that $m \leq n$. In this representation, every entry is a boolean (we're in binary!) $a[0]$ is the "least-significant-bit" (the "one's place").

Your goal is to return their sum in binary (return it as a `boolean[]` as well, in the same format). For this problem you may **only** use the boolean operators $\neg, \wedge, \vee$ in finding the sum of these two binary integers.

You are free to use java-like syntax (including java structures like loops) along with propositional logic notation. But you may not use any operator other than $\neg, \wedge, \vee$ to combine/alter the booleans.

(for a greater challenge, limit yourself to only two of these operators, and consider why you cannot solve this task using only one of these operators).

Hint: Consider breaking this problem down. First, consider the maximal possible size of the output array. Then, isolate a single "column" in the addition, and consider what information you need to find the corresponding output digit. Then, try adding small binary numbers by hand and seeing how you would construct these addition operations using only boolean operators. Good luck!

9. Feedback [1 point]

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment (excluding any extra credit questions, if applicable)? Report your estimate to the nearest hour.
- Which problem did you spend the most time on?
- Any other feedback for us?