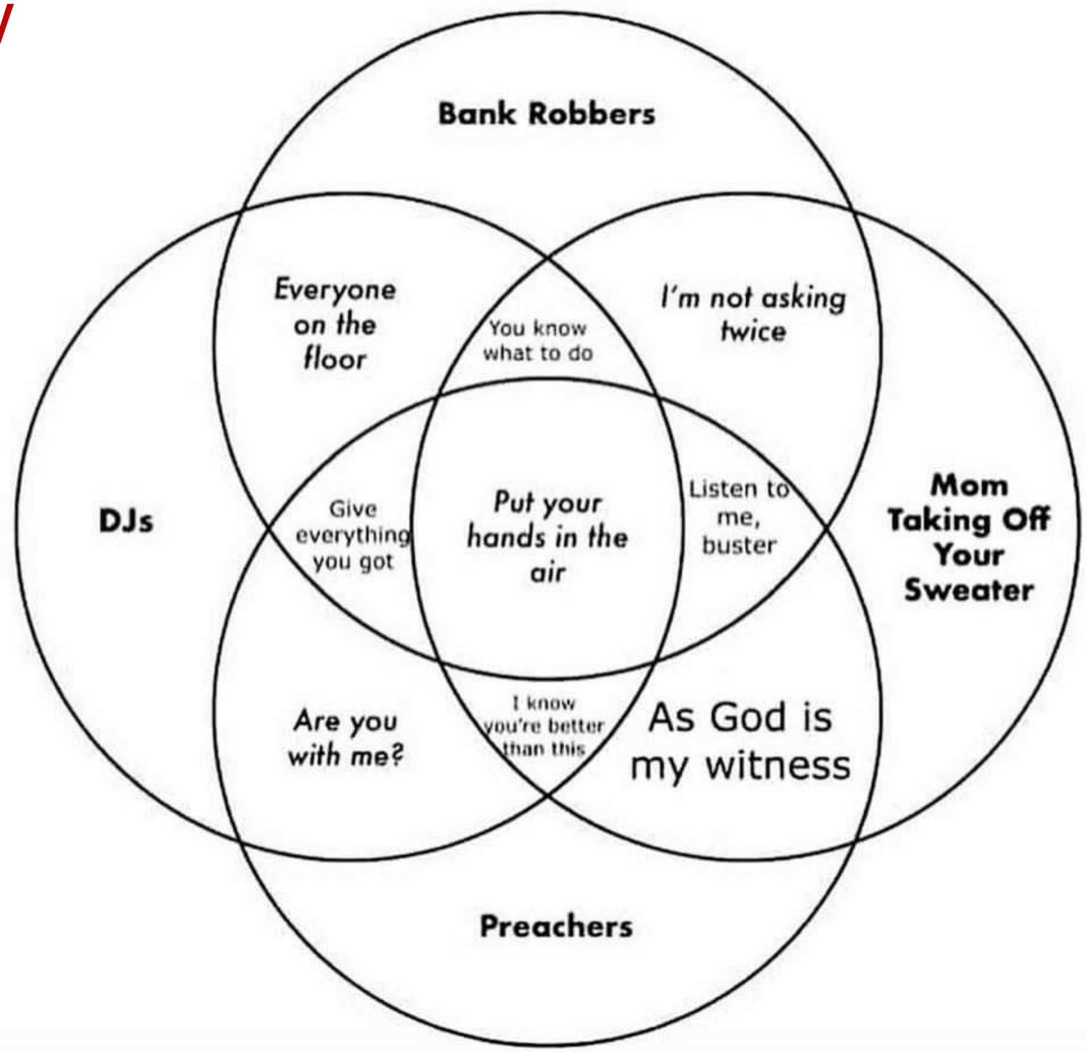
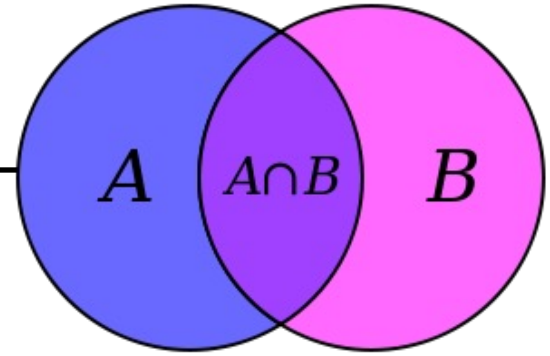
CSE 311: Foundations of Computing

Topic 6: Set Theory



Sets



Sets are collections of objects called elements.

Write $a \in B$ **to say that** a **is an element of set** B**,**
and $a \notin B$ **to say that it is not.**

Some simple examples
$A = \{1\}$
$B = \{1, 3, 2\}$
$C = \{\square, 1\}$
$D = \{\{17\}, 17\}$
$E = \{1, 2, 7, \text{cat}, \text{dog}, \varnothing, \alpha\}$

Some Common Sets

$\mathbb{N}$ is the set of **Natural Numbers;** $\mathbb{N} = \{0, 1, 2, \ldots\}$

$\mathbb{Z}$ is the set of **Integers**; $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Q}$ is the set of **Rational Numbers**; e.g. ½, -17, 32/48

$\mathbb{R}$ is the set of **Real Numbers**; e.g. 1, -17, 32/48, $\pi, \sqrt{2}$

[n] is the set **{1, 2, …, n}** when **n** is a natural number

$\varnothing$ = **{}** is the **empty set**; the *only* set with no elements

Sets can be elements of other sets

For example

$A = \{\{1\},\{2\},\{1,2\},\varnothing\}$

$B = \{1,2\}$

Then $B \in A$.

Definitions

- A and B are *equal* if they have the same elements

$$A = B \;:=\; \forall\, x\, (x \in A \leftrightarrow x \in B)$$

- A is a *subset* of B if every element of A is also in B

$$A \subseteq B \;:=\; \forall\, x\, (x \in A \rightarrow x \in B)$$

Definitions

- A and B are *equal* if they have the same elements

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$$A \subseteq B \ := \ \forall x\,(x \in A \rightarrow x \in B)$$

- Notes:

$$(\boldsymbol{A} = \boldsymbol{B}) \ \equiv \ (\boldsymbol{A} \subseteq \boldsymbol{B}) \wedge (\boldsymbol{B} \subseteq \boldsymbol{A})$$

$A \supseteq B$ means $B \subseteq A$

$A \subset B$ means $A \subseteq B$

Definition: Equality

A and B are *equal* if they have the same elements

$$A = B \ := \ \forall x \, (x \in A \leftrightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}
D = {4, 3, 3}
E = {3, 4, 3}
F = {4, {3}}

Which sets are equal?

Definition: Subset

A is a *subset* of B if every element of A is also in B

$$A \subseteq B \;:=\; \forall x\,(x \in A \rightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}

<u>QUESTIONS</u>

$A \subseteq B$?
$C \subseteq B$?
$\varnothing \subseteq A$?

Definition: Subset

A is a *subset* of B if every element of A is also in B

$$A \subseteq B \ := \ \forall\, x\, (x \in A \rightarrow x \in B)$$

Note the domain restriction!

We will use a shorthand restriction to a set

$\forall x{\in}A\ (P(x))$ means $\forall x\ (x \in A \rightarrow P(x))$

Restricting all quantified variables improves *clarity*

Sets & Logic

Building Sets from Predicates

Every set S **defines a predicate** $P(x) :=$ "$x \in S$"

We can also define a set from a predicate P**:**

$$\boxed{S \;:=\; \{x : P(x)\}}$$

S **= the set of all** x **for which** $P(x)$ **is true**

$$\boxed{S \;:=\; \{x \in U : P(x)\} \;=\; \{x : (x \in U) \land P(x)\}}$$

Inference Rules on Sets

S := {x : P(x)}

When a set is defined this way,
we can reason about it using its definition:

1.	$x \in S$	Given
2.	$P(x)$	Def of S
	…	
8.	$P(y)$	
9.	$y \in S$	Def of S

This will be our **only** inference rule for sets!

Proofs About Sets

$A := \{x : P(x)\}$ $B := \{x : Q(x)\}$

Suppose we want to prove $A \subseteq B$.

We have a definition of subset:

$$A \subseteq B := \forall x\,(x \in A \rightarrow x \in B)$$

We need to show that is definition holds

Proofs About Sets

$A := \{x : P(x)\}$

$B := \{x : Q(x)\}$

8. $\forall x\,(x \in A \to x \in B)$ **??**
9. $A \subseteq B$ **Def of Subset: 8**

Proofs About Sets

$A := \{x : P(x)\}$ $B := \{x : Q(x)\}$

Let x be arbitrary

1.	$x \in A \rightarrow x \in B$	??
2.	$\forall x\,(x \in A \rightarrow x \in B)$	Intro $\forall$: 1
3.	$A \subseteq B$	Def of Subset: 2

Proofs About Sets

$A := \{x : P(x)\}$ $B := \{x : Q(x)\}$

Let x be arbitrary

1.1. $x \in A$ Assumption

1.9. $x \in B$??

1. $x \in A \rightarrow x \in B$ Direct Proof
2. $\forall x\,(x \in A \rightarrow x \in B)$ Intro $\forall$: 1
3. $A \subseteq B$ Def of Subset: 2

Proofs About Sets

$A := \{x : P(x)\}$ $B := \{x : Q(x)\}$

Let x be arbitrary

1.1.	$x \in A$	Assumption
1.2.	$P(x)$	Def of A
1.8.	$Q(x)$	
1.9.	$x \in B$	Def of B

1.	$x \in A \to x \in B$	Direct Proof
2.	$\forall x\,(x \in A \to x \in B)$	Intro $\forall$: 1
3.	$A \subseteq B$	Def of Subset: 2

Proofs About Sets

$A := \{x : P(x)\}$ $B := \{x : Q(x)\}$

Prove that $A \subseteq B$**.**

Proof: Let x be an arbitrary object.

Suppose that $x \in A$. By definition of A, this means $P(x)$.

…

Thus, we have $Q(x)$. By definition of B, this means $x \in B$.

Since x was arbitrary, we have shown, by definition, that $A \subseteq B$.

Operations on Sets

Set Operations

$$A \cup B := \{\, x : (x \in A) \vee (x \in B)\}$$ **Union**

$$A \cap B := \{\, x : (x \in A) \wedge (x \in B)\}$$ **Intersection**

$$A \setminus B := \{\, x : (x \in A) \wedge (x \notin B)\}$$ **Set Difference**

A = {1, 2, 3}
B = {3, 5, 6}
C = {3, 4}

QUESTIONS

Using A, B, C and set operations, make...

[6] =

{3} =

{1,2} =

More Set Operations

$$A \oplus B := \{ x : (x \in A) \oplus (x \in B)\}$$

Symmetric Difference

$$\overline{A} = A^C := \{ x : x \in U \wedge x \notin A \}$$

(with respect to universe U)

Complement

A = {1, 2, 3}
B = {1, 2, 4, 6}
Universe:
U = {1, 2, 3, 4, 5, 6}

$A \oplus B$ = {3, 4, 6}
$\overline{A}$ = {4,5,6}

De Morgan's Laws

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall x\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Since x was arbitrary, we have shown, by definition, that $(A \cup B)^C = A^C \cap B^C$.

Proof technique:
To show C = D show
$x \in C \rightarrow x \in D$ and
$x \in D \rightarrow x \in C$

De Morgan's Laws

Formally, prove $\forall x\,(x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

1. Let x be arbitrary
   - 2.1. $x \in (A \cup B)^C$ — Assumption
   - ...
   - 2.3. $x \in A^C \cap B^C$
2. $x \in (A \cup B)^C \rightarrow x \in A^C \cap B^C$ — Direct Proof
   - 3.1. $x \in A^C \cap B^C$ — Assumption
   - ...
   - 3.3. $x \in (A \cup B)^C$
3. $x \in A^C \cap B^C \rightarrow x \in (A \cup B)^C$ — Direct Proof
4. $(x \in (A \cup B)^C \rightarrow x \in A^C \cap B^C) \wedge (x \in A^C \cap B^C \rightarrow x \in (A \cup B)^C)$ — Intro $\wedge$: 2, 3
5. $x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C$ — Biconditional: 4
6. $\forall x\,(x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$ — Intro $\forall$: 1-5

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$.

...

Thus, we have $x \in A^C \cap B^C$.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$.

…

Thus, we have $x \in A^C \cap B^C$.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \vee x \in B)$.

…

Thus, we have $x \in A^C \cap B^C$.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \vee x \in B)$.

…

Thus, $x \in A^C$ and $x \in B^C$, so we we have $x \in A^C \cap B^C$ by the definition of intersection.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \vee x \in B)$.

…

Thus, $\neg(x \in A)$ and $\neg(x \in B)$, so $x \in A^C$ and $x \in B^C$ by the definition of compliment, and we can see that $x \in A^C \cap B^C$ by the definition of intersection.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \vee x \in B)$, or equivalently $\neg(x \in A) \wedge \neg(x \in B)$ by De Morgan's law. Thus, we have $x \in A^C$ and $x \in B^C$ by the definition of compliment, and we can see that $x \in A^C \cap B^C$ by the definition of intersection.

Proof technique:
To show C = D show
$x \in \mathrm{C} \rightarrow x \in \mathrm{D}$ and
$x \in \mathrm{D} \rightarrow x \in \mathrm{C}$

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$.... Then, $x \in A^C \cap B^C$.

Suppose $x \in A^C \cap B^C$. Then, by the definition of intersection, we have $x \in A^C$ and $x \in B^C$. That is, we have $\neg(x \in A) \land \neg(x \in B)$, which is equivalent to $\neg(x \in A \lor x \in B)$ by De Morgan's law. The last is equivalent to $\neg(x \in A \cup B)$, by the definition of union, so we have shown $x \in (A \cup B)^C$, by the definition of complement.

Proofs About Set Equality

A lot of *repetitive* work to show $\longrightarrow$ and $\longleftarrow$.

Do we have a way to prove $\leftrightarrow$ directly?

Recall that $A \equiv B$ and $(A \leftrightarrow B) \equiv T$ are the same

We can use an equivalence chain to prove that a biconditional holds.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall x\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

The stated biconditional holds since:

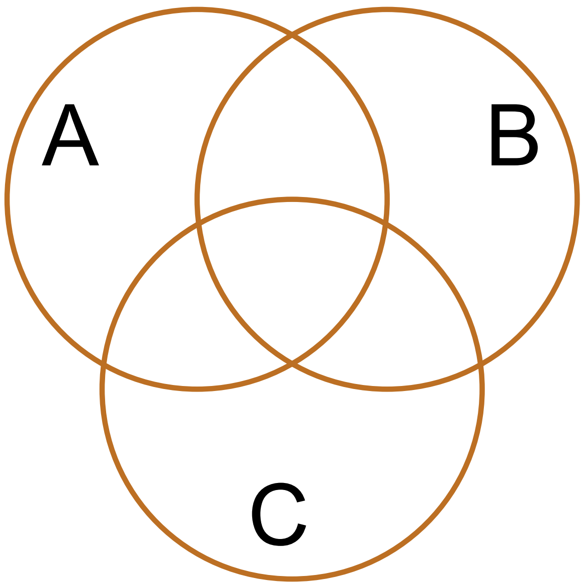
$x \in (A \cup B)^C$	$\equiv \neg(x \in A \cup B)$	Def of Comp
	$\equiv \neg(x \in A \vee x \in B)$	Def of Union
	$\equiv \neg(x \in A) \wedge \neg(x \in B)$	De Morgan
	$\equiv x \in A^C \wedge x \in B^C$	Def of Comp
	$\equiv x \in A^C \cap B^C$	Def of Intersection

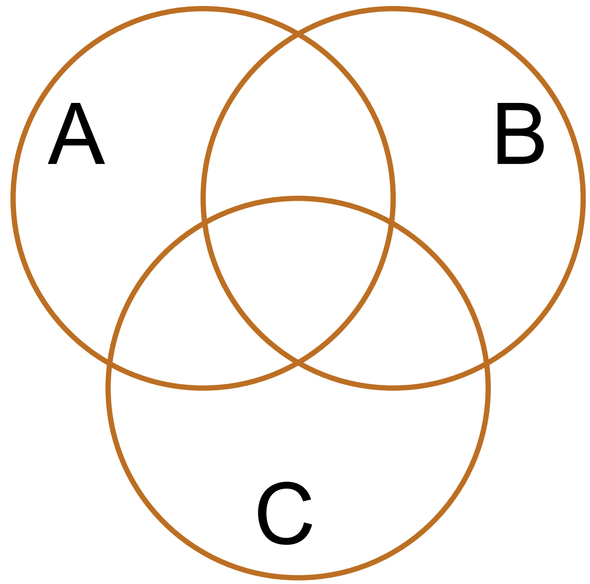
Chains of equivalences are often easier to read like this rather than as English text

Since x was arbitrary, we have shown the sets are equal. ■

Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$





It's Propositional Logic again!

The Meta Theorem Template

Meta-Theorem: Translate any Propositional Logic equivalence into "=" relationship between sets by replacing $\cup$ with $\vee$, $\cap$ with $\wedge$, and $\cdot^{C}$ with $\neg$.

"Proof": Let x be an arbitrary object.

The stated bi-condition holds since:

$x \in$ left side $\equiv$ replace set ops with propositional logic

$\equiv$ apply Propositional Logic equivalence

$\equiv$ replace propositional logic with set ops

$\equiv x \in$ right side

Since x was arbitrary, we have shown the sets are equal. ■

Power Set

- Power Set of a set A = set of all subsets of A

$$\mathcal{P}(A) := \{B : B \subseteq A\}$$

- e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class

$\mathcal{P}$(Days)=?

$\mathcal{P}(\varnothing)$=?

Power Set

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$$\mathcal{P}(A) := \{B : B \subseteq A\}$$

- e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class

$\mathcal{P}(\text{Days})=\{\{M, W, F\}, \{M, W\}, \{M, F\}, \{W, F\}, \{M\}, \{W\}, \{F\}, \varnothing\}$

$\mathcal{P}(\varnothing)=?$

Power Set

- Power Set of a set A = set of all subsets of A

$$\mathcal{P}(A) := \{B : B \subseteq A\}$$

- e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class

$\mathcal{P}(\text{Days})=\{\{\text{M, W, F}\}, \{\text{M, W}\}, \{\text{M, F}\}, \{\text{W, F}\}, \{\text{M}\}, \{\text{W}\}, \{\text{F}\}, \varnothing\}$

$\mathcal{P}(\varnothing)=\{\varnothing\} \neq \varnothing$

Cartesian Product

$$A \times B := \{x : \exists a \in A\ \exists b \in B\ (x = (a, b))\ \}$$

$\mathbb{R} \times \mathbb{R}$ is the real plane. You've seen ordered pairs before.

These are just for arbitrary sets.

$\mathbb{Z} \times \mathbb{Z}$ is "the set of all pairs of integers"

If A = {1, 2}, B = {a, b, c}, then A × B = {(1,a), (1,b), (1,c),
(2,a), (2,b), (2,c)}.

Cartesian Product

$$A \times B := \{x : \exists a \in A\ \exists b \in B\ (x = (a, b))\}$$

$\mathbb{R} \times \mathbb{R}$ is the real plane. You've seen ordered pairs before.

These are just for arbitrary sets.

$\mathbb{Z} \times \mathbb{Z}$ is "the set of all pairs of integers"

If $A = \{1, 2\}$, $B = \{a, b, c\}$, then $A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$.

What is $A \times \emptyset$?

Cartesian Product

$$A \times B := \{x : \exists a \in A\ \exists b \in B\ (x = (a, b))\}$$

$\mathbb{R} \times \mathbb{R}$ is the real plane. You've seen ordered pairs before.

These are just for arbitrary sets.

$\mathbb{Z} \times \mathbb{Z}$ is "the set of all pairs of integers"

If A = {1, 2}, B = {a, b, c}, then A × B = {(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)}.

$$\boldsymbol{A} \times \emptyset = \{(\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \boldsymbol{A} \land \boldsymbol{b} \in \emptyset\} = \{(\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \boldsymbol{A} \land \mathbf{F}\} = \emptyset$$

Russell's Paradox

$$S := \{x : x \notin x\}$$

Suppose that $S \in S$...

Russell's Paradox

$$S := \{x : x \notin x\}$$

Suppose that $S \in S$. Then, by the definition of S, $S \notin S$, but that's a contradiction.

Suppose that $S \notin S$. Then, by the definition of S, $S \in S$, but that's a contradiction too.

This is reminiscent of the truth value of the statement "This statement is false."