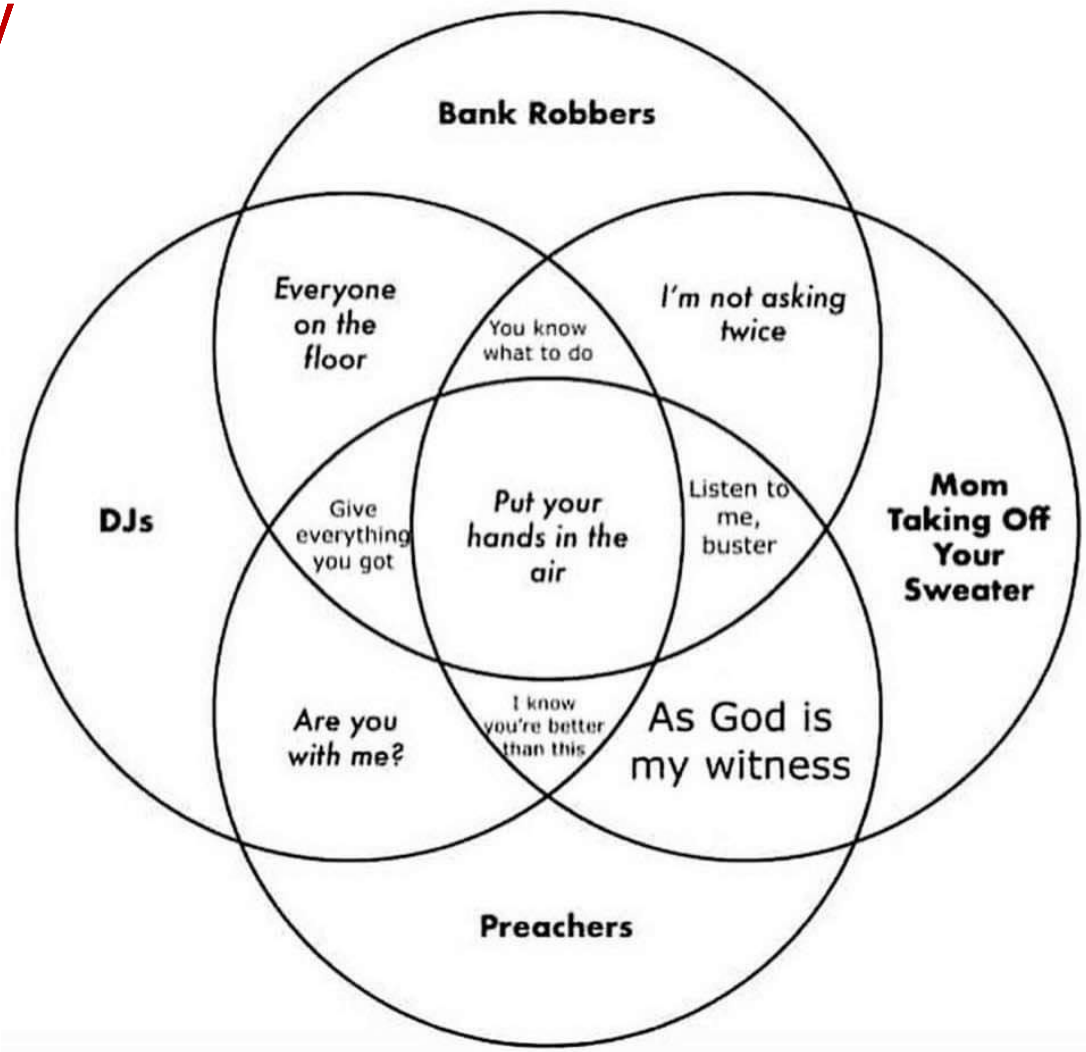
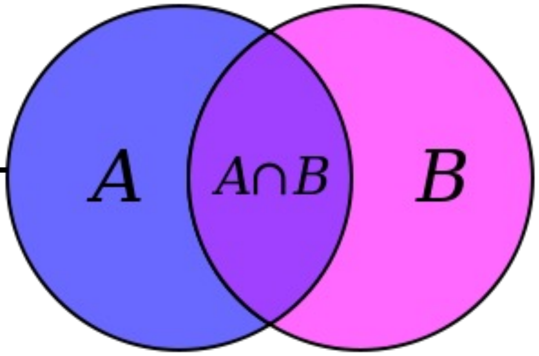
CSE 311: Foundations of Computing

Topic 6: Set Theory



Sets



Sets are collections of objects called **elements.**

Write $a \in B$ to say that a is an element of set B, and $a \notin B$ to say that it is not.

Some simple examples

$A = \{1\}$

$B = \{1, 3, 2\}$

$C = \{\square, 1\}$

$D = \{\{17\}, 17\}$

$E = \{1, 2, 7, \text{cat}, \text{dog}, \varnothing, \alpha\}$

Some Common Sets

$\mathbb{N}$ is the set of **Natural Numbers;** $\mathbb{N} = \{0, 1, 2, \ldots\}$

$\mathbb{Z}$ is the set of **Integers**; $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Q}$ is the set of **Rational Numbers**; e.g. ½, -17, 32/48

$\mathbb{R}$ is the set of **Real Numbers**; e.g. 1, -17, 32/48, $\pi, \sqrt{2}$

[n] is the set **{1, 2, …, n}** when **n** is a natural number

$\varnothing$ = **{}** is the **empty set**; the *only* set with no elements

Sets can be elements of other sets

For example

$A = \{\{1\},\{2\},\{1,2\},\emptyset\}$

$B = \{1,2\}$

Then $B \in A$.

Definitions

- A and B are *equal* if they have the same elements

$$A = B \; := \; \forall x\, (x \in A \leftrightarrow x \in B)$$

- A is a *subset* of B if every element of A is also in B

$$A \subseteq B \; := \; \forall x\, (x \in A \rightarrow x \in B)$$

Definitions

- A and B are *equal* if they have the same elements

$$A = B \;:=\; \forall x\,(x \in A \leftrightarrow x \in B)$$

- A is a *subset* of B if every element of A is also in B

$$A \subseteq B \;:=\; \forall x\,(x \in A \rightarrow x \in B)$$

- Notes: $\boldsymbol{(A = B) \equiv (A \subseteq B) \wedge (B \subseteq A)}$

$A \supseteq B$ means $B \subseteq A$

$A \subset B$ means $A \subseteq B$

Definition: Equality

A and B are *equal* if they have the same elements

$$A = B \ := \ \forall x\,(x \in A \leftrightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}
D = {4, 3, 3}
E = {3, 4, 3}
F = {4, {3}}

Which sets are equal?

Definition: Subset

A is a *subset* of B if every element of A is also in B

$$A \subseteq B \;:=\; \forall x\,(x \in A \rightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}

<u>QUESTIONS</u>

$A \subseteq B$?
$C \subseteq B$?
$\varnothing \subseteq A$?

Definition: Subset

A is a *subset* of B if every element of A is also in B

$$A \subseteq B \ := \ \forall x \, (x \in A \rightarrow x \in B)$$

Note the domain restriction!

We will use a shorthand restriction to a set

$\forall x \in A \, (P(x))$ means $\forall x \, (x \in A \rightarrow P(x))$

Restricting all quantified variables improves *clarity*

Sets & Logic

Building Sets from Predicates

Every set S defines a predicate $P(x) := \text{“}x \in S\text{”}$

We can also define a set from a predicate P:

$$\boxed{S := \{x : P(x)\}}$$

S **= the set of all** x **for which** $P(x)$ **is true**

$$\boxed{S := \{x \in U : P(x)\} = \{x : (x \in U) \land P(x)\}}$$

Inference Rules on Sets

$$S := \{x : P(x)\}$$

When a set is defined this way,
we can reason about it using its definition:

1.	$x \in S$	Given
2.	$P(x)$	Def of S
	…	
8.	$P(y)$	
9.	$y \in S$	Def of S

This will be our **only** inference rule for sets!

Proofs About Sets

$A := \{x : P(x)\}$ $\quad$ $B := \{x : Q(x)\}$

Suppose we want to prove $A \subseteq B$.

We have a definition of subset:

$$A \subseteq B := \forall x\, (x \in A \rightarrow x \in B)$$

We need to show that is definition holds

Proofs About Sets

$A := \{x : P(x)\}$

$B := \{x : Q(x)\}$

8. $\forall x\,(x \in A \rightarrow x \in B)$??
9. $A \subseteq B$ Def of Subset: 8

Proofs About Sets

$A := \{x : P(x)\}$ $\quad$ $B := \{x : Q(x)\}$

Let x be arbitrary

1.1. $x \in A$ Assumption

1.9. $x \in B$??

1. $x \in A \rightarrow x \in B$ Direct Proof
2. $\forall x\, (x \in A \rightarrow x \in B)$ Intro $\forall$: 1
3. $A \subseteq B$ Def of Subset: 2

Proofs About Sets

$A := \{x : P(x)\}$ $B := \{x : Q(x)\}$

Let x be arbitrary

1.1.	$x \in A$	Assumption
1.2.	$P(x)$	Def of **A**
1.8.	$Q(x)$	
1.9.	$x \in B$	Def of **B**

1. $x \in A \rightarrow x \in B$ Direct Proof
2. $\forall x\,(x \in A \rightarrow x \in B)$ Intro $\forall$: 1
3. $A \subseteq B$ Def of Subset: 2

Proofs About Sets

$A := \{x : P(x)\}$ $B := \{x : Q(x)\}$

Prove that $A \subseteq B$.

Proof: Let x be an arbitrary object.

Suppose that $x \in A$. By definition of A, this means $P(x)$.

…

Thus, we have $Q(x)$. By definition of B, this means $x \in B$.

Since x was arbitrary, we have shown, by definition, that $A \subseteq B$.

Operations on Sets

Set Operations

$$A \cup B := \{\, x : (x \in A) \vee (x \in B)\}$$ **Union**

$$A \cap B := \{\, x : (x \in A) \wedge (x \in B)\}$$ **Intersection**

$$A \setminus B := \{\, x : (x \in A) \wedge (x \notin B)\}$$ **Set Difference**

A = {1, 2, 3}
B = {3, 5, 6}
C = {3, 4}

QUESTIONS

Using A, B, C and set operations, make…

[6] =

{3} =

{1,2} =

More Set Operations

$$A \oplus B := \{\, x : (x \in A) \oplus (x \in B)\}$$

Symmetric Difference

$$\bar{A} = A^C := \{\, x : x \in U \wedge x \notin A \,\}$$

(with respect to universe U)

Complement

A = {1, 2, 3}
B = {1, 2, 4, 6}
Universe:
U = {1, 2, 3, 4, 5, 6}

$A \oplus B$ = {3, 4, 6}
$\bar{A}$ = {4,5,6}

De Morgan's Laws

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Since x was arbitrary, we have shown, by definition, that $(A \cup B)^C = A^C \cap B^C$.

Proof technique:
To show C = D show
$x \in C \rightarrow x \in D$ and
$x \in D \rightarrow x \in C$

De Morgan's Laws

Formally, prove $\forall x\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

1. Let x be arbitrary
    2.1. $x \in (A \cup B)^C$ Assumption

    ...

    2.3. $x \in A^C \cap B^C$
2. $x \in (A \cup B)^C \rightarrow x \in A^C \cap B^C$ Direct Proof
    3.1. $x \in A^C \cap B^C$ Assumption

    ...

    3.3. $x \in (A \cup B)^C$
3. $x \in A^C \cap B^C \rightarrow x \in (A \cup B)^C$ Direct Proof
4. $(x \in (A \cup B)^C \rightarrow x \in A^C \cap B^C) \wedge (x \in A^C \cap B^C \rightarrow x \in (A \cup B)^C)$ Intro $\wedge$: 2, 3
5. $x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C$ Biconditional: 4
6. $\forall x\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$ Intro $\forall$: 1-5

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$.

...

Thus, we have $x \in A^C \cap B^C$.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall x\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$.

…

Thus, we have $x \in A^C \cap B^C$.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \lor x \in B)$.

…

Thus, we have $x \in A^C \cap B^C$.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \text{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \vee x \in B)$.

…

Thus, $x \in A^C$ and $x \in B^C$, so we we have $x \in A^C \cap B^C$ by the definition of intersection.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \vee x \in B)$.

…

Thus, $\neg(x \in A)$ and $\neg(x \in B)$, so $x \in A^C$ and $x \in B^C$ by the definition of compliment, and we can see that $x \in A^C \cap B^C$ by the definition of intersection.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \vee x \in B)$, or equivalently $\neg(x \in A) \wedge \neg(x \in B)$ by De Morgan's law. Thus, we have $x \in A^C$ and $x \in B^C$ by the definition of compliment, and we can see that $x \in A^C \cap B^C$ by the definition of intersection.

Proof technique:
To show C = D show
$x \in \mathrm{C} \rightarrow x \in \mathrm{D}$ and
$x \in \mathrm{D} \rightarrow x \in \mathrm{C}$

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$.... Then, $x \in A^C \cap B^C$.

Suppose $x \in A^C \cap B^C$. Then, by the definition of intersection, we have $x \in A^C$ and $x \in B^C$. That is, we have $\neg(x \in A) \wedge \neg(x \in B)$, which is equivalent to $\neg(x \in A \vee x \in B)$ by De Morgan's law. The last is equivalent to $\neg(x \in A \cup B)$, by the definition of union, so we have shown $x \in (A \cup B)^C$, by the definition of complement.

Proofs About Set Equality

A lot of *repetitive* work to show $\longrightarrow$ and $\longleftarrow$.

Do we have a way to prove $\leftrightarrow$ directly?

Recall that $A \equiv B$ and $(A \leftrightarrow B) \equiv T$ are the same

We can use an equivalence chain to prove that a biconditional holds.

De Morgan’s Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall x\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

The stated biconditional holds since:

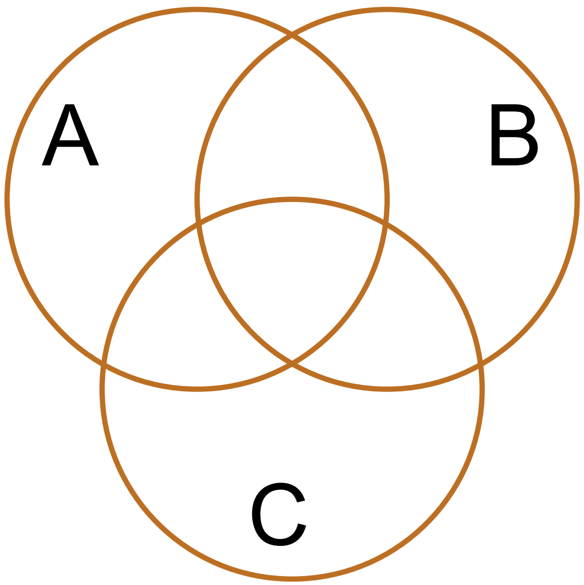
$x \in (A \cup B)^C$	$\equiv \neg(x \in A \cup B)$	Def of Comp
	$\equiv \neg(x \in A \vee x \in B)$	Def of Union
	$\equiv \neg(x \in A) \wedge \neg(x \in B)$	De Morgan
	$\equiv x \in A^C \wedge x \in B^C$	Def of Comp
	$\equiv x \in A^C \cap B^C$	Def of Union

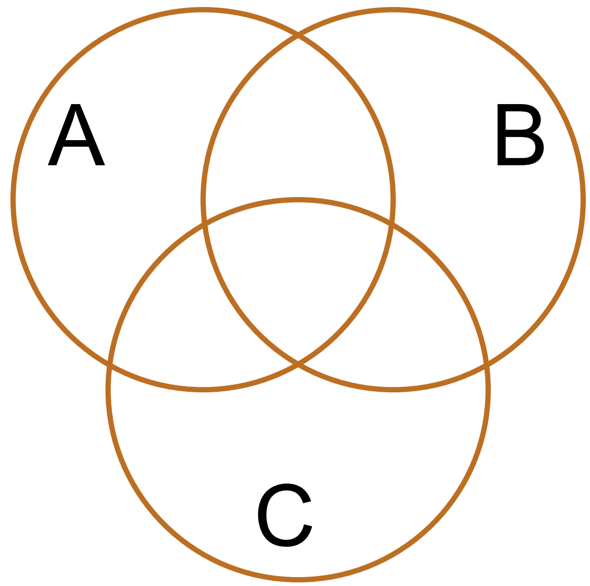
Chains of equivalences are often easier to read like this rather than as English text

Since x was arbitrary, we have shown the sets are equal. ■

Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$





It's Propositional Logic again!

The Meta Theorem Template

Meta-Theorem: Translate any Propositional Logic equivalence into "=" relationship between sets by replacing ∪ with ∨, ∩ with ∧, and $\cdot^{C}$ with ¬.

"Proof": Let x be an arbitrary object.

The stated bi-condition holds since:

x ∈ left side ≡ replace set ops with propositional logic

≡ apply Propositional Logic equivalence

≡ replace propositional logic with set ops

≡ x ∈ right side

Since x was arbitrary, we have shown the sets are equal. ■

Power Set

- Power Set of a set A = set of all subsets of A

$$\mathcal{P}(A) := \{B : B \subseteq A\}$$

- e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class

$\mathcal{P}$(Days)=?

$\mathcal{P}(\varnothing)$=?

Power Set

- **Power Set of a set A = set of all subsets of A**

$$\mathcal{P}(A) := \{B : B \subseteq A\}$$

- **e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class**

$$\mathcal{P}(\text{Days})=\{\{M, W, F\}, \{M, W\}, \{M, F\}, \{W, F\}, \{M\}, \{W\}, \{F\}, \varnothing\}$$

$$\mathcal{P}(\varnothing)=?$$

Power Set

- Power Set of a set A = set of all subsets of A

$$\mathcal{P}(A) := \{B : B \subseteq A\}$$

- e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class

$\mathcal{P}(\text{Days})=\{\{M, W, F\}, \{M, W\}, \{M, F\}, \{W, F\}, \{M\}, \{W\}, \{F\}, \varnothing\}$

$\mathcal{P}(\varnothing)=\{\varnothing\} \neq \varnothing$

Cartesian Product

$$A \times B := \{x : \exists a \in A\ \exists b \in B\ (x = (a, b))\ \}$$

$\mathbb{R} \times \mathbb{R}$ is the real plane. You've seen ordered pairs before.

These are just for arbitrary sets.

$\mathbb{Z} \times \mathbb{Z}$ is "the set of all pairs of integers"

If A = {1, 2}, B = {a, b, c}, then A $\times$ B = {(1,a), (1,b), (1,c),
(2,a), (2,b), (2,c)}.

Cartesian Product

$$A \times B := \{x : \exists a \in A\ \exists b \in B\ (x = (a, b))\}$$

$\mathbb{R} \times \mathbb{R}$ is the real plane. You've seen ordered pairs before.

These are just for arbitrary sets.

$\mathbb{Z} \times \mathbb{Z}$ is "the set of all pairs of integers"

If A = {1, 2}, B = {a, b, c}, then A × B = {(1,a), (1,b), (1,c),
(2,a), (2,b), (2,c)}.

What is $A \times \emptyset$?

Cartesian Product

$$A \times B := \{x : \exists a \in A\ \exists b \in B\ (x = (a, b))\}$$

$\mathbb{R} \times \mathbb{R}$ **is the real plane. You've seen ordered pairs before.**

These are just for arbitrary sets.

$\mathbb{Z} \times \mathbb{Z}$ **is "the set of all pairs of integers"**

If A = {1, 2}, B = {a, b, c}, then A × B = {(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)}.

$$\boldsymbol{A} \times \emptyset = \{(\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \boldsymbol{A} \wedge \boldsymbol{b} \in \emptyset\} = \{(\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \boldsymbol{A} \wedge \mathbf{F}\} = \emptyset$$

Russell's Paradox

$$S := \{x : x \notin x\}$$

Suppose that $S \in S$...

Russell’s Paradox

$$S := \{x : x \notin x\}$$

Suppose that $S \in S$. Then, by the definition of S, $S \notin S$, but that’s a contradiction.

Suppose that $S \notin S$. Then, by the definition of S, $S \in S$, but that’s a contradiction too.

This is reminiscent of the truth value of the statement “This statement is false.”