Quiz Section 10: Cardinality and Computability

Task 1 – Review: Subsets

Consider the S defined recursively as follows:

> **Basis:** $1 \in S$.
> **Recursive Step:** If $x \in S$ and $a \in \{0, 1\}$, then $xa \in S$.

and the set T of strings that start with a 1, which is defined formally as follows:

$$T := \{x \in \{0,1\}^* : \exists y \in \{0,1\}^* \, (x = 1 \bullet y)\}$$

Use structural induction to prove that $\forall x \in S \, (x \in T)$.

Task 2 – Irregularity

a) Let $\Sigma = \{0, 1\}$. Prove that $\{0^n 1^n 0^n \ : \ n \geqslant 0\}$ is not regular.

b) Let $\Sigma = \{0, 1, 2\}$. Prove that $\{0^n (12)^m \ : \ n \geqslant m \geqslant 0\}$ is not regular.

Task 3 – Cardinality

a) You are a pirate. You begin in a square on a 2D grid that is infinite in all directions. In other words, wherever you are, you may move up, down, left, or right. Some single square on the infinite grid has treasure on it. Find a way to ensure you find the treasure in finitely many moves.

b) Prove that $\{3x \ : \ x \in \mathbb{N}\}$ is countable.

c) Prove that the set of irrational numbers is uncountable.
Hint: Use the fact that the rationals are countable and that the reals are uncountable.

d) Prove that $\mathcal{P}(\mathbb{N})$ is uncountable.

Task 4 – Countable Unions

a) Show that $\mathbb{N} \times \mathbb{N}$ is countable.

Hint: How did we show that the rationals were countable?

b) Show that the countable union of countable sets is countable. That is, given a collection of sets $S_1, S_2, S_2, \ldots$ such that S_i is countable for all $i \in \mathbb{N}$, show that

$$S = S_1 \cup S_2 \cup \cdots = \{x : x \in S_i \text{ for some } i\}$$

is countable.

Hint: Find a way of labeling the elements and see if you can apply the previous part to construct an onto function from $\mathbb{N}$ to S.

Task 5 – RE to NFA

Convert the regular expression "$(11 \cup (01)^*)00$" to an NFA using the algorithm from lecture. You may skip adding ε-transitions for concatenation if they are obviously unnecessary, but otherwise, you should *precisely* follow the construction from lecture.

Task 6 – DFA/REGEXP/CFG

For each of the following languages, construct a DFA, Regular Expression, and CFG for it.

(a) $A = \{w \in \{0,1\}^* :$ the number of 0's minus the number of 1's in w is divisible by 3$\}$.