CSE 311 Section 10

Final Review

Announcements & Reminders

- HW8
  - Due **Friday** 5/31 @ 11:59
- Final Exam
  - Monday, 6/3 12:30-2:20pm KNE120

- Final Review Session
  - Today @ 5:00-7:00pm SIG134

- Course Evaluations are out!
  - Please consider taking 10 minutes to complete both section and course evaluations!

Structural Induction Problem 1

Consider the S defined recursively as follows:

Basis: $1 \in S$.
Recursive Step: If $x \in S$ and $a \in \{0,1\}$, then $xa \in S$.

and the set T of strings that start with a 1, which is defined formally as follows:

$$T := \{x \in \{0,1\}^* : \exists y \in \{0,1\}^* \, (x = 1 \bullet y)\}$$

Use structural induction to prove that $\forall x \in S \, (x \in T)$.

Consider the S defined recursively as follows:

Basis: $1 \in S$.
Recursive Step: If $x \in S$ and $a \in \{0, 1\}$, then $xa \in S$.

and the set T of strings that start with a 1, which is defined formally as follows:

$$T := \{x \in \{0,1\}^* : \exists y \in \{0,1\}^* \, (x = 1 \bullet y)\}$$

Use structural induction to prove that $\forall x \in S \, (x \in T)$.

Define $P(x)$ to be the claim $x \in T$, i.e., that x can be written as $1 \bullet y$ for some $y \in \{0,1\}^*$.

Consider the S defined recursively as follows:

Basis: $1 \in S$.
Recursive Step: If $x \in S$ and $a \in \{0,1\}$, then $xa \in S$.

and the set T of strings that start with a 1, which is defined formally as follows:

$$T := \{x \in \{0,1\}^* : \exists y \in \{0,1\}^* \, (x = 1 \bullet y)\}$$

Use structural induction to prove that $\forall x \in S \, (x \in T)$.

Define $P(x)$ to be the claim $x \in T$, i.e., that x can be written as $1 \bullet y$ for some $y \in \{0,1\}^*$.
We will show that $P(x)$ is true for every $x \in S$ by structural induction.

Consider the S defined recursively as follows:

Basis: $1 \in S$.
Recursive Step: If $x \in S$ and $a \in \{0,1\}$, then $xa \in S$.

and the set T of strings that start with a 1, which is defined formally as follows:

$$T := \{x \in \{0,1\}^* : \exists y \in \{0,1\}^* \, (x = 1 \bullet y)\}$$

Use structural induction to prove that $\forall x \in S \, (x \in T)$.

Define $P(x)$ to be the claim $x \in T$, i.e., that x can be written as $1 \bullet y$ for some $y \in \{0,1\}^*$. We will show that $P(x)$ is true for every $x \in S$ by structural induction.

Base Case (1). We can see that $1 = 1 \bullet \varepsilon$, which shows that $1 \in T$.

Consider the S defined recursively as follows:

Basis: $1 \in S$.
Recursive Step: If $x \in S$ and $a \in \{0,1\}$, then $xa \in S$.

and the set T of strings that start with a 1, which is defined formally as follows:

$$T := \{x \in \{0,1\}^* : \exists y \in \{0,1\}^* \, (x = 1 \bullet y)\}$$

Use structural induction to prove that $\forall x \in S \, (x \in T)$.

Define $P(x)$ to be the claim $x \in T$, i.e., that x can be written as $1 \bullet y$ for some $y \in \{0,1\}^*$. We will show that $P(x)$ is true for every $x \in S$ by structural induction.

Base Case (1). We can see that $1 = 1 \bullet \varepsilon$, which shows that $1 \in T$.

Inductive Hypothesis. Suppose that $P(x)$ holds for an arbitrary $x \in S$.

Consider the S defined recursively as follows:

Basis: $1 \in S$.
Recursive Step: If $x \in S$ and $a \in \{0,1\}$, then $xa \in S$.

and the set T of strings that start with a 1, which is defined formally as follows:

$$T := \{x \in \{0,1\}^* : \exists y \in \{0,1\}^* \, (x = 1 \bullet y)\}$$

Use structural induction to prove that $\forall x \in S \, (x \in T)$.

Define $P(x)$ to be the claim $x \in T$, i.e., that x can be written as $1 \bullet y$ for some $y \in \{0,1\}^*$. We will show that $P(x)$ is true for every $x \in S$ by structural induction.

Base Case (1). We can see that $1 = 1 \bullet \varepsilon$, which shows that $1 \in T$.

Inductive Hypothesis. Suppose that $P(x)$ holds for an arbitrary $x \in S$.

Inductive Step. Let $a \in \{0,1\}$ be arbitrary. We must show that $xa \in T$.

Consider the S defined recursively as follows:

Basis: $1 \in S$.
Recursive Step: If $x \in S$ and $a \in \{0, 1\}$, then $xa \in S$.

and the set T of strings that start with a 1, which is defined formally as follows:

$$T := \{x \in \{0,1\}^* : \exists y \in \{0,1\}^* \, (x = 1 \bullet y)\}$$

Use structural induction to prove that $\forall x \in S \, (x \in T)$.

Define $P(x)$ to be the claim $x \in T$, i.e., that x can be written as $1 \bullet y$ for some $y \in \{0,1\}^*$. We will show that $P(x)$ is true for every $x \in S$ by structural induction.

Base Case (1). We can see that $1 = 1 \bullet \varepsilon$, which shows that $1 \in T$.

Inductive Hypothesis. Suppose that $P(x)$ holds for an arbitrary $x \in S$.

Inductive Step. Let $a \in \{0, 1\}$ be arbitrary. We must show that $xa \in T$.

To do so, first note that the Inductive Hypothesis tells us that there is some $y \in \{0,1\}^*$ such that $x = 1 \bullet y$.

Consider the S defined recursively as follows:

Basis: $1 \in S$.
Recursive Step: If $x \in S$ and $a \in \{0, 1\}$, then $xa \in S$.

and the set T of strings that start with a 1, which is defined formally as follows:

$$T := \{x \in \{0, 1\}^* : \exists y \in \{0, 1\}^* \, (x = 1 \bullet y)\}$$

Use structural induction to prove that $\forall x \in S \, (x \in T)$.

Define $P(x)$ to be the claim $x \in T$, i.e., that x can be written as $1 \bullet y$ for some $y \in \{0, 1\}^*$. We will show that $P(x)$ is true for every $x \in S$ by structural induction.

Base Case (1). We can see that $1 = 1 \bullet \varepsilon$, which shows that $1 \in T$.

Inductive Hypothesis. Suppose that $P(x)$ holds for an arbitrary $x \in S$.

Inductive Step. Let $a \in \{0, 1\}$ be arbitrary. We must show that $xa \in T$.

To do so, first note that the Inductive Hypothesis tells us that there is some $y \in \{0, 1\}^*$ such that $x = 1 \bullet y$. Thus, we can see that $x = (1 \bullet y)a = 1 \bullet (ya)$ by the definition of string concatenation, which shows that $P(xa)$ holds.

Consider the S defined recursively as follows:

Basis: $1 \in S$.
Recursive Step: If $x \in S$ and $a \in \{0, 1\}$, then $xa \in S$.

and the set T of strings that start with a 1, which is defined formally as follows:

$$T := \{x \in \{0, 1\}^* : \exists y \in \{0, 1\}^* \, (x = 1 \bullet y)\}$$

Use structural induction to prove that $\forall x \in S \, (x \in T)$.

Define $P(x)$ to be the claim $x \in T$, i.e., that x can be written as $1 \bullet y$ for some $y \in \{0, 1\}^*$. We will show that $P(x)$ is true for every $x \in S$ by structural induction.

Base Case (1). We can see that $1 = 1 \bullet \varepsilon$, which shows that $1 \in T$.

Inductive Hypothesis. Suppose that $P(x)$ holds for an arbitrary $x \in S$.

Inductive Step. Let $a \in \{0, 1\}$ be arbitrary. We must show that $xa \in T$.

To do so, first note that the Inductive Hypothesis tells us that there is some $y \in \{0, 1\}^*$ such that $x = 1 \bullet y$. Thus, we can see that $x = (1 \bullet y)a = 1 \bullet (ya)$ by the definition of string concatenation, which shows that $P(xa)$ holds.

Conclusion. $P(x)$ holds for all $x \in S$ by structural induction.

Irregularity

Irregularity Template

Claim: L is an irregular language.

Proof: Suppose, for the sake of contradiction, that L is regular. Then there is a DFA M such that M accepts exactly L.

Let S = [TODO] (S is an infinite set of strings)
Because the DFA is finite, there are two (different) strings x, y in S such that x and y go to the same state when read by M. [TODO] (We don't get to choose x, y, but we can describe them based on that set S we just defined)

Consider the string z = [TODO] (We do get to choose z depending on x, y)

Since x, y led to the same state and M is deterministic, xz and yz will also lead to the same state q in M. Observe that xz = [TODO], so $xz \in L$ but yz = [TODO], so $yz \notin L$. Since q is can be only one of an accept or reject state, M does not actually recognize L. That's a contradiction!

Therefore, L is an irregular language.

Irregularity Example from Lecture

Claim: $\{0^k1^k : k \geq 0\}$ is an irregular language.

Proof: Suppose, for the sake of contradiction, that $L = \{0^k1^k : k \geq 0\}$ is regular. Then there is a DFA M such that M accepts exactly L.

Let $S = \{0^k : k \geq 0\}$
Because the DFA is finite, there are two (different) strings x, y in S such that x and y go to the same state when read by M. Since both are in S, $x = 0^a$ for some integer $a \geq 0$, and $y = 0^b$ for some integer $b \geq 0$, with $a \neq b$.

Consider the string $z = 1^a$.

Since x, y led to the same state and M is deterministic, xz and yz will also lead to the same state q in M. Observe that $xz = 0^a1^a$, so $xz \in L$ but $yz = 0^b1^a$, so $yz \notin L$. Since q is can be only one of an accept or reject state, M does not actually recognize L. That's a contradiction!

Therefore, L is an irregular language.

Problem 2 – Irregularity

a) Let $\Sigma = \{0, 1\}$. Prove that $\{0^n1^n0^n : n \geq 0\}$ is not regular.

b) Let $\Sigma = \{0, 1, 2\}$. Prove that $\{0^n(12)^m : n \geq m \geq 0\}$ is not regular.

Work on this problem with the people around you.

Problem 2 – Irregularity

(a) Let $\Sigma = \{0, 1\}$. Prove that $\{0^n1^n0^n : n \geq 0\}$ is not regul

Claim: $\{0^n1^n0^n : n \geq 0\}$ is an irregular language.

Proof: Suppose, for the sake of contradiction, that $L = \{0^n1^n0^n : n \geq 0\}$ is regular. Then there is a DFA M such that M accepts exactly L.

Let S = [TODO]
Because the DFA is finite, there are two (different) strings x, y in S such that x and y go to the same state when read by M. [TODO] .

Consider the string z = [TODO] .

Since x, y led to the same state and M is deterministic, xz and yz will also lead to the same state q in M. Observe that xz = [TODO] , so $xz \in L$ but yz = [TODO] , so $yz \notin L$. Since q is can be only one of an accept or reject state, M does not actually recognize L. That's a contradiction!

Therefore, L is an irregular language.

Problem 2 – Irregularity

(a) Let $\Sigma = \{0, 1\}$. Prove that $\{0^n1^n0^n : n \geq 0\}$ is not regul

Claim: $\{0^n1^n0^n : n \geq 0\}$ is an irregular language.

Proof: Suppose, for the sake of contradiction, that $L = \{0^n1^n0^n : n \geq 0\}$ is regular. Then there is a DFA M such that M accepts exactly L.

Let S = $\{0^n1^n : n \geq 0\}$
Because the DFA is finite, there are two (different) strings x, y in S such that x and y go to the same state when read by M. [TODO] .

Consider the string z = [TODO] .

Since x, y led to the same state and M is deterministic, xz and yz will also lead to the same state q in M. Observe that xz = [TODO] , so $xz \in L$ but yz = [TODO] , so $yz \notin L$. Since q is can be only one of an accept or reject state, M does not actually recognize L. That's a contradiction!

Therefore, L is an irregular language.

Problem 2 – Irregularity

(a) Let $\Sigma = \{0, 1\}$. Prove that $\{0^n1^n0^n : n \geq 0\}$ is not regul

Claim: $\{0^n1^n0^n : n \geq 0\}$ is an irregular language.

Proof: Suppose, for the sake of contradiction, that $L = \{0^n1^n0^n : n \geq 0\}$ is regular. Then there is a DFA M such that M accepts exactly L.

Let $S = \{0^n1^n : n \geq 0\}$
Because the DFA is finite, there are two (different) strings x, y in S such that x and y go to the same state when read by M. Since both are in S, $x = 0^a1^a$ for some integer $a \geq 0$, and $y = 0^b1^b$ for some integer $b \geq 0$, with $a \neq b$.

Consider the string z = [TODO] .

Since x, y led to the same state and M is deterministic, xz and yz will also lead to the same state q in M. Observe that xz = [TODO] , so $xz \in L$ but yz = [TODO] , so $yz \notin L$. Since q is can be only one of an accept or reject state, M does not actually recognize L. That's a contradiction!

Therefore, L is an irregular language.

Problem 2 – Irregularity

(a) Let $\Sigma = \{0, 1\}$. Prove that $\{0^n1^n0^n : n \geq 0\}$ is not regul

Claim: $\{0^n1^n0^n : n \geq 0\}$ is an irregular language.

Proof: Suppose, for the sake of contradiction, that $L = \{0^n1^n0^n : n \geq 0\}$ is regular. Then there is a DFA M such that M accepts exactly L.

Let $S = \{0^n1^n : n \geq 0\}$
Because the DFA is finite, there are two (different) strings x, y in S such that x and y go to the same state when read by M. Since both are in S, $x = 0^a1^a$ for some integer $a \geq 0$, and $y = 0^b1^b$ for some integer $b \geq 0$, with $a \neq b$.

Consider the string $z = 0^a$.

Since x, y led to the same state and M is deterministic, xz and yz will also lead to the same state q in M. Observe that xz = [TODO] , so $xz \in L$ but yz = [TODO] , so $yz \notin L$. Since q is can be only one of an accept or reject state, M does not actually recognize L. That's a contradiction!

Therefore, L is an irregular language.

Problem 2 – Irregularity (a) Let $\Sigma = \{0, 1\}$. Prove that $\{0^n1^n0^n : n \geq 0\}$ is not regul

Claim: $\{0^n1^n0^n : n \geq 0\}$ is an irregular language.

Proof: Suppose, for the sake of contradiction, that $L = \{0^n1^n0^n : n \geq 0\}$ is regular. Then there is a DFA M such that M accepts exactly L.

Let $S = \{0^n1^n : n \geq 0\}$
Because the DFA is finite, there are two (different) strings x, y in S such that x and y go to the same state when read by M. Since both are in S, $x = 0^a1^a$ for some integer $a \geq 0$, and $y = 0^b1^b$ for some integer $b \geq 0$, with $a \neq b$.

Consider the string $z = 0^a$.

Since x, y led to the same state and M is deterministic, xz and yz will also lead to the same state q in M. Observe that $xz = 0^a1^a0^a$, so $xz \in L$ but $yz = 0^b1^b0^a$, so $yz \notin L$. Since q is can be only one of an accept or reject state, M does not actually recognize L. That's a contradiction!

Therefore, L is an irregular language.

Problem 2 – Irregularity

(b) Let $\Sigma = \{0, 1, 2\}$. Prove that $\{0^n(12)^m : n \geq m \geq 0\}$ is not regular.

Claim: $\{0^n(12)^m : n \geq m \geq 0\}$ is an irregular language.

Proof: Suppose, for the sake of contradiction, that $L = \{0^n(12)^m : n \geq m \geq 0\}$ is regular. Then there is a DFA M such that M accepts exactly L.

Let S = [TODO]
Because the DFA is finite, there are two (different) strings x, y in S such that x and y go to the same state when read by M. [TODO] .

Consider the string z = [TODO] .

Since x, y led to the same state and M is deterministic, xz and yz will also lead to the same state q in M. Observe that xz = [TODO] , so $xz \in L$ but yz = [TODO] , so $yz \notin L$. Since q is can be only one of an accept or reject state, M does not actually recognize L. That's a contradiction!

Therefore, L is an irregular language.

Problem 2 – Irregularity

(b) Let $\Sigma = \{0, 1, 2\}$. Prove that $\{0^n(12)^m : n \geq m \geq 0\}$ is not regular.

Claim: $\{0^n(12)^m : n \geq m \geq 0\}$ is an irregular language.

Proof: Suppose, for the sake of contradiction, that $L = \{0^n(12)^m : n \geq m \geq 0\}$ is regular. Then there is a DFA M such that M accepts exactly L.

Let $S = \{0^n : n \geq 0\}$
Because the DFA is finite, there are two (different) strings x, y in S such that x and y go to the same state when read by M. [TODO]

Consider the string z = [TODO] .

Since x, y led to the same state and M is deterministic, xz and yz will also lead to the same state q in M. Observe that xz = [TODO] , so $xz \in L$ but yz = [TODO] , so $yz \notin L$. Since q is can be only one of an accept or reject state, M does not actually recognize L. That's a contradiction!

Therefore, L is an irregular language.

Problem 2 – Irregularity

(b) Let $\Sigma = \{0, 1, 2\}$. Prove that $\{0^n(12)^m : n \geq m \geq 0\}$ is not regular.

Claim: $\{0^n(12)^m : n \geq m \geq 0\}$ is an irregular language.

Proof: Suppose, for the sake of contradiction, that $L = \{0^n(12)^m : n \geq m \geq 0\}$ is regular. Then there is a DFA M such that M accepts exactly L.

Let $S = \{0^n : n \geq 0\}$
Because the DFA is finite, there are two (different) strings x, y in S such that x and y go to the same state when read by M. Since both are in S, $x = 0^a$ for some integer $a \geq 0$, and $y = 0^b$ for some integer $b \geq 0$, with $a > b$.

Consider the string z = [TODO] .

Since x, y led to the same state and M is deterministic, xz and yz will also lead to the same state q in M. Observe that xz = [TODO] , so $xz \in L$ but yz = [TODO] , so $yz \notin L$. Since q is can be only one of an accept or reject state, M does not actually recognize L. That's a contradiction!

Therefore, L is an irregular language.

Problem 2 – Irregularity

(b) Let $\Sigma = \{0, 1, 2\}$. Prove that $\{0^n(12)^m : n \geq m \geq 0\}$ is not regular.

Claim: $\{0^n(12)^m : n \geq m \geq 0\}$ is an irregular language.

Proof: Suppose, for the sake of contradiction, that $L = \{0^n(12)^m : n \geq m \geq 0\}$ is regular. Then there is a DFA M such that M accepts exactly L.

Let S = $\{0^n : n \geq 0\}$
Because the DFA is finite, there are two (different) strings x, y in S such that x and y go to the same state when read by M. Since both are in S, $x = 0^a$ for some integer $a \geq 0$, and $y = 0^b$ for some integer $b \geq 0$, with $a > b$.

Consider the string z = $(12)^a$.

Since x, y led to the same state and M is deterministic, xz and yz will also lead to the same state q in M. Observe that xz = [TODO] , so $xz \in L$ but yz = [TODO] , so $yz \notin L$. Since q is can be only one of an accept or reject state, M does not actually recognize L. That's a contradiction!

Therefore, L is an irregular language.

Problem 2 – Irregularity

(b) Let $\Sigma = \{0, 1, 2\}$. Prove that $\{0^n(12)^m : n \geq m \geq 0\}$ is not regular.

Claim: $\{0^n(12)^m : n \geq m \geq 0\}$ is an irregular language.

Proof: Suppose, for the sake of contradiction, that $L = \{0^n(12)^m : n \geq m \geq 0\}$ is regular. Then there is a DFA M such that M accepts exactly L.

Let $S = \{0^n : n \geq 0\}$

Because the DFA is finite, there are two (different) strings x, y in S such that x and y go to the same state when read by M. Since both are in S, $x = 0^a$ for some integer $a \geq 0$, and $y = 0^b$ for some integer $b \geq 0$, with $a > b$.

Consider the string $z = (12)^a$.

Since x, y led to the same state and M is deterministic, xz and yz will also lead to the same state q in M. Observe that $xz = 0^a(12)^a$, so $xz \in L$ but $yz = 0^b(12)^a$, so $yz \notin L$. Since q is can be only one of an accept or reject state, M does not actually recognize L. That's a contradiction!

Therefore, L is an irregular language.

Task 3c Countability

Countability

Prove that the set of irrational numbers is uncountable.
Hint: Use the fact that the rationals are countable and that the reals are uncountable.

Countability

Prove that the set of irrational numbers is uncountable.
Hint: Use the fact that the rationals are countable and that the reals are uncountable.

We first prove that the union of two countable sets is countable. Consider two arbitrary countable sets C_1 and C_2. We can enumerate $C_1 \cup C_2$ by mapping even natural numbers to C_1 and odd natural numbers to C_2.

Countability

Prove that the set of irrational numbers is uncountable.
Hint: Use the fact that the rationals are countable and that the reals are uncountable.

We first prove that the union of two countable sets is countable. Consider two arbitrary countable sets C_1 and C_2. We can enumerate $C_1 \cup C_2$ by mapping even natural numbers to C_1 and odd natural numbers to C_2.

Now, assume that the set of irrationals is countable. Then the reals would be countable, since the reals are the union of the irrationals (countable by assumption) and the rationals (countable). However, we have already shown that the reals are uncountable, which is a contradiction. Therefore, our assumption that the set of irrationals is countable is false, and the irrationals must be uncountable.

Task 5 or 6!

That’s All, Folks!

Thanks for coming to section this week!
Any questions?