Quiz Section 5: Set Theory, Induction

Review

Set theory:

- $A \backslash B = \{x : x \in A \wedge x \notin B\}$. Or, equivalently, $x \in A \backslash B \leftrightarrow x \in A \wedge x \notin B$.
- $A \times B = \{(a, b) : a \in A,\ b \in B\}$. Or, equivalently, $(a, b) \in A \times B \leftrightarrow a \in A \wedge b \in B$.
- $\mathcal{P}(A) = \{B : B \subseteq A\}$. Or, equivalently, $B \in \mathcal{P}(A) \leftrightarrow B \subseteq A$.

5 Steps to an Induction Proof: To prove $\forall n \in \mathbb{N}\ P(n)$ (or equivalently $\forall n \geqslant 0\ P(n)$ for $n \in \mathbb{Z}$).

1. "Let $P(n)$ be ⟨fill in⟩. We will show that $P(n)$ is true for every $n \in \mathbb{N}$ (or equivalently integer $n \geqslant 0$) by induction."
2. "Base Case:" Prove $P(0)$
3. "Inductive Hypothesis: Suppose $P(k)$ is true for some arbitrary integer $k \geqslant 0$"
4. "Inductive Step:" Prove that $P(k+1)$ is true.

   Use the goal to figure out what you need.
   Make sure you are using I.H. and point out where you are using it.
   (Don't assume $P(k+1)$!)
5. "Conclusion: The claim follows by induction"

Task 1 – How Many Elements?

For each of these, how many elements are in the set? If the set has infinitely many elements, say ∞.

a) $A = \{1, 2, 3, 2\}$

b) $B = \{\{\}, \{\{\}\}, \{\{\}, \{\}\}, \{\{\}, \{\}, \{\}\}, \ldots\}$

c) $D = \varnothing$

d) $E = \{\varnothing\}$

e) $C = A \times (B \cup \{7\})$

f) $G = \mathcal{P}(\{\varnothing\})$

Task 2 – Set Replay

Let A, B, and C be sets. Prove each of the following claims **formally**. Then, translate to **English**.

a) $A \backslash B \subseteq A \cup C$ (Try it on Cozy here: http://bit.ly/S053A)

b) $(A \backslash B) \backslash C \subseteq A \backslash C$ (Try it on Cozy here: http://bit.ly/S053B)

Task 3 – Set Equality

Let A and B be sets. Consider the claim: $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$.

State what the claim becomes when you unroll the definition of "=" sets. Then, following the Meta Theorem template, write an **English proof** that the claim holds.

Task 4 – Congruence Solver

a) Find the multiplicative inverse y of 7 mod 33. That is, find y such that $7y \equiv 1 \pmod{33}$. You should use the extended Euclidean Algorithm. Your answer should be in the range $0 \leqslant y < 33$.

b) Now, solve $7z \equiv_{33} 2$ for all of its integer solutions z.

Task 5 – Power Sets

Let A and B be sets. Write a **formal proof** that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ follows from $A \subseteq B$. Then, translate your proof to English.

Task 6 – Induction with Equality

a) Define the triangle numbers as $\triangle_n = 0 + 1 + 2 + \cdots + n$, where $n \in \mathbb{N}$.

We already saw in class that $\triangle_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$$

b) For every $n \in \mathbb{N}$, define S_n to be the sum of the squares of the natural numbers up to n, or

$$S_n = 0^2 + 1^2 + \cdots n^2.$$

For all $n \in \mathbb{N}$, prove that $S_n = \frac{1}{6}n(n+1)(2n+1)$.

Task 7 – Induction with Divides

Prove that $2 \mid 3^n - 1$ for all $n \geqslant 0$ by induction.