Problem Set 6

Due: Wednesday, May 15th by 11:00pm

Instructions

Solutions submission. You must submit your solution via Gradescope. In particular:

– Submit a *single* PDF file containing your solutions to Tasks 1–6 (and optionally 7). Follow the prompt on Gradescope to link tasks to your pages.

Task 1 – Barking Up the Strong Tree [20 pts]

The function $f(m)$ is defined for $m \in \mathbb{N}$ recursively as follows:

$$\begin{aligned} f(0) &= 1 \\ f(1) &= 2 \\ f(m) &= f(m-1) + 2 \cdot f(m-2) && \text{if } m \geqslant 2 \end{aligned}$$

Use strong induction to prove that

$$\forall n \in \mathbb{N}\,(f(n) = 2^n)$$

Write an English proof, following the template given in lecture.

Task 2 – Live Strong and Prosper [20 pts]

The function $g(m)$ is defined for integers $m \geqslant 1$ recursively as follows:

$$\begin{aligned} g(1) &= 0 \\ g(2l) &= 1 + g(l) && \text{where } l \geqslant 1 \text{ is an integer} \\ g(2l+1) &= 1 + g(l) && \text{where } l \geqslant 1 \text{ is an integer} \end{aligned}$$

The first line gives the definition of g for $m = 1$, the second line gives the definition for even m, and the third line gives the definition for odd $m \geqslant 3$. Since these three cases are mutually exclusive and exhaustive, they define g completely.

Use strong induction on n to show that $2^{g(n)} \leqslant n$ for all integers $n \geqslant 1$.

Write an *English* proof, following the template given in lecture.

Recall the definition of lists of numbers from lecture:

Basis Step: $\texttt{nil} \in$ **List**
Recursive Step: for any $a \in \mathbb{Z}$, if $L \in$ **List**, then $a :: L \in$ **List**.

For example, the list $[1, 2, 3]$ would be created recursively from the empty list as $1 :: (2 :: (3 :: \texttt{nil}))$. We will consider "::" to associate to the right, so $1 :: 2 :: 3 :: \texttt{nil}$ means the same thing.

The next two problems use three recursively-defined functions. The first is len, which calculates the length of the list. It is defined recursively as follows:

$$\begin{aligned} \text{len}(\texttt{nil}) &:= 0 \\ \text{len}(a :: L) &:= 1 + \text{len}(L) \quad \forall a \in \mathbb{Z}, \forall L \in \textbf{List} \end{aligned}$$

The second function, concat, which concatenates two lists into a single list, is defined by:

$$\begin{aligned} \text{concat}(\texttt{nil}, R) &:= R & \forall R \in \textbf{List} \\ \text{concat}(a :: L, R) &:= a :: \text{concat}(L, R) & \forall a \in \mathbb{Z}, \forall L, R \in \textbf{List} \end{aligned}$$

For example, we get $\text{concat}(1 :: 2 :: \texttt{nil}, 3 :: \texttt{nil}) = 1 :: 2 :: 3 :: \texttt{nil}$ from these definitions.

The third function, positives, which returns only the positive numbers in the list, is defined by:

$$\begin{aligned} \text{positives}(\texttt{nil}) &:= \texttt{nil} & & \\ \text{positives}(a :: L) &:= \text{positives}(L) & \text{if } a \leq 0 & \quad \forall a \in \mathbb{Z}, \forall L \in \textbf{List} \\ \text{positives}(a :: L) &:= a :: \text{positives}(L) & \text{if } a > 0 & \quad \forall a \in \mathbb{Z}, \forall L \in \textbf{List} \end{aligned}$$

For example, from these definitions, we get $\text{positives}(-1 :: 2 :: -3 :: \texttt{nil}) = 2 :: \texttt{nil}$.

Task 3 – List Me By a Mile [20 pts]

a) Write a chain of equalities, citing the appropriate definitions, showing that

$$\text{concat}(1 :: 2 :: \texttt{nil}, 3 :: \texttt{nil}) = 1 :: 2 :: 3 :: \texttt{nil}$$

b) Write a chain of equalities, citing the appropriate definitions, showing that

$$\text{positives}(1 :: -2 :: 3 :: \texttt{nil}) = 1 :: 3 :: \texttt{nil}$$

c) Use structural induction to prove that

$$\forall L \in \textbf{List}\ (\text{len}(\text{positives}(L)) \leq \text{len}(L))$$

Task 4 – List Me, List Me, Now You Gotta Kiss Me [18 pts]

Let $R, S \in$ **List**. Use structural induction on L to prove that

$$\forall L \in \textbf{List}\ (\text{concat}(\text{concat}(L, R), S) = \text{concat}(L, \text{concat}(R, S)))$$

i.e., that concat is associative.

Task 5 – A Few of My Favorite Strings [12 pts]

For each of the following, write a recursive definition of the set of strings satisfying the given properties. *Briefly* justify that your solution is correct.

a) Binary strings that start with 1 and have odd length.

b) Binary strings where every 1 is immediately followed by a 00.

c) Binary strings with an odd number of 1s.

Task 6 – Donny, You're Out of Your Element[1] [18 pts]

Let A and B be the following sets:

$$A := \{n \in \mathbb{Z} : 3 \mid n\}$$
$$B := \{n \in \mathbb{Z} : (3 \mid n) \vee (6 \mid n)\}$$

Write an **English proof** that $A = B$.

Do not use the Meta Theorem template. Instead, prove the necessary biconditional by proving each implication independently.

Task 7 – Extra Credit: Stone By the Company He Keeps [0 pts]

Consider an infinite sequence of positions $1, 2, 3, \dots$ and suppose we have a stone at position 1 and another stone at position 2. In each step, we choose one of the stones and move it according to the following rule: Say we decide to move the stone at position i; if the other stone is not at any of the positions $i+1, i+2, \dots, 2i$, then it goes to $2i$, otherwise it goes to $2i+1$.

For example, in the first step, if we move the stone at position 1, it will go to 3 and if we move the stone at position 2 it will go to 4. Note: no matter how we move the stones, they will never be at the same position.

Use induction to prove that, for any given positive integer n, it is possible to move one of the stones to position n. For example, if $n = 7$ first we move the stone at position 1 to 3. Then, we move the stone at position 2 to 5 Finally, we move the stone at position 3 to 7.

[1] No punny shtuff — Nihilist