Problem Set 5

Due: Friday, May 3rd by 11:00pm

Instructions

Solutions submission. You must submit your solution via Gradescope. In particular:

– Submit a *single* PDF file containing your solutions to Task 1(b), 2(b), 3, 5, 6 (and optionally 7). Follow the prompt on Gradescope to link tasks to your pages.

– The instructions for submitting Task 1(a), 2(a), and 4 appear below those individual problems.

Task 1 – The Dude Divides [16 pts]

Let A and B be the following sets:

$$A := \{n \in \mathbb{Z} : 6 \mid n + 4\}$$
$$B := \{n \in \mathbb{Z} : 2 \mid n - 4\}$$

Now, consider the following claim:

$$A \subseteq B$$

a) Write a **formal proof** that the claim holds.

In Cozy, you can replace "$x \in A$" by "$6 \mid x+4$" with "`defof A`" and the reverse with "`undef A`".[1] You can replace "$A \subseteq B$" by "$\forall x\,(x \in A \to x \in B)$" with "`defof subset {A} {B}`" and the reverse with "`undef subset {A} {B}`". (Here, you must also say *which* sets you are comparing. Just saying "`defof subset`" wouldn't be enough information for Cozy.)

> Submit and check your formal proof here:
>
> http://cozy.cs.washington.edu
>
> You **must also** include your solution (as a screenshot, typeset LaTeX, or rewritten by hand) in the PDF you submit to **Gradescope**.

b) Translate your formal proof to an **English proof**.

Keep in mind that your proof will be read by a *human*, not a computer, so you should explain the algebra steps in more detail, whereas some of the predicate logic steps (e.g., Elim $\exists$) can be skipped.

[1] Those commands should work as stated when reasoning backward. For forward reasoning, you need to add the line number at the end, e.g., `defof A 1.1.1`.

Task 2 – Feelin' Proovy [16 pts]

Let A and B be sets. Now, consider the following claim:

$$(A\backslash B)\backslash C \cup (B \cap C) \subseteq A \cup B$$

a) Write a **formal proof** that the claim holds.

In Cozy, you can replace "$x \in A \cup B$" by "$x \in A \vee x \in B$" with "`defof cup {A} {B}`" and the reverse with "`undef cup {A} {B}`". Likewise, the commands for $x \in A \cap B$, $x \in A\backslash B$, and $x \in A^C$ would be "`defof cap {A} {B}`", "`defof \ {A} {B}`", and "`defof ~ {A}`", respectively.

Hint: After unrolling the definitions, the fact you know should be a "$\vee$" (from unrolling "$\cup$"). It may be easiest to prove the required conclusion holds *by cases* using that "$\vee$".

Submit and check your formal proof here:

http://cozy.cs.washington.edu

You **must also** include your solution (as a screenshot, typeset LaTeX, or rewritten by hand) in the PDF you submit to **Gradescope**.

b) Translate your formal proof to an **English proof**.

Follow the same rules as in Task 1.

Task 3 – Parmesian, Romano, and Meta [8 pts]

Let A, B, and C be sets. For each of the following claims, write an **English proof** that the claim holds by following the Meta Theorem *template* given in lecture.

Note: In your equivalence chain, you can skip steps showing commutativity or associativity, as long as each step is easy to follow.

a) $(A \backslash B) \cup \overline{C} = (A \cup \overline{C}) \backslash (B \cap C)$

b) $B \backslash (A \backslash (C \cup B)) = (A \cup \overline{A}) \backslash \overline{B}$

Task 4 – Our Finest Power [14 pts]

Let the domain of discourse be sets of integers, and let A, B, and C be some sets.
Write a **formal proof** that $\mathcal{P}(A) \cap \mathcal{P}(C) \subseteq \mathcal{P}(B \cap C)$ follows from $A \subseteq B$.

Your proof will need the definition of $\mathcal{P}$ in addition to the other set operations. In Cozy, you can replace "$x \in \mathcal{P}(A)$" by its definition, "$x \subseteq A$", with "`defof PowerSet`" and the reverse with "`undef`". In Cozy's syntax, you write $x \in \mathcal{P}(A)$ as "`x in power(A)`".

Submit and check your formal proof here: http://cozy.cs.washington.edu You can make as many attempts as needed to find a correct answer.

Task 5 – Alien Induction [20 pts]

Prove, by induction, that

$$\sum_{i=0}^{n} (3i + 1) = \frac{3n^2 + 5n + 2}{2}$$

holds for all $n \in \mathbb{N}$.

Write an *English* proof, following the template given in lecture.

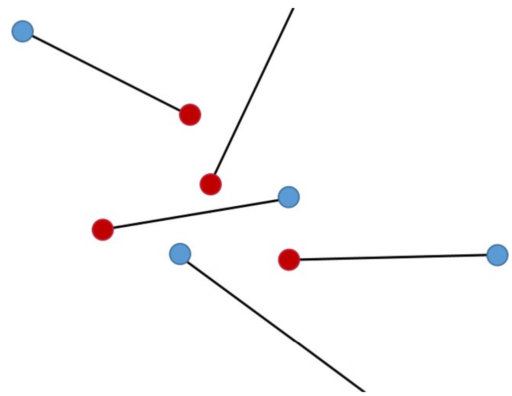
Task 6 – Super-Colliding Super Inductor [20 pts]

Prove, by induction, that $2n^3 + n$ is divisible by 3 for any $n \in \mathbb{N}$.

Write an *English* proof, following the template given in lecture.

Task 7 – Extra Credit: Match Me If You Can [0 pts]

In this problem, you will show that given n red points and n blue points in the plane such that no three points lie on a common line, it is possible to draw line segments between red-blue pairs so that all the pairs are matched and none of the line segments intersect. Assume that there are n red and n blue points fixed in the plane.



A *matching* M is a collection of n line segments connecting distinct red-blue pairs. The *total length* of a matching M is the sum of the lengths of the line segments in M. Say that a matching M is *minimal* if there is no matching with a smaller total length.

Let IsMinimal(M) be the predicate that is true precisely when M is a minimal matching. Let HasCrossing(M) be the predicate that is true precisely when there are two line segments in M that cross each other.

Give an argument in English explaining why there must be at least one matching M so that IsMinimal(M) is true, i.e.

$$\exists M \text{IsMinimal}(M))$$

Give an argument in English explaining why

$$\forall M(\text{HasCrossing}(M) \rightarrow \neg\text{IsMinimal}(M))$$

Then, use the two results above to give a proof of the statement:

$$\exists M \neg\text{HasCrossing}(M).$$