CSE 311: Foundations of Computing I Spring 2024

Problem Set 2

Due: Wednesday, April 10th by 11:00pm

Instructions

Solutions submission. You must submit your solution via Gradescope. In particular:

- Submit a *single* PDF file containing the solution to all tasks *except* for Task 2. Follow the prompt on Gradescope to link tasks to your pages.
- The instructions for submitting Task 2 appear below that problem.

Task 1 – A Bit Long in the Truth [15 pts]

For each of the following pairs of propositions, use truth tables to determine whether they are equivalent.

Include the full truth table and state whether they are equivalent. (In principle, only one row is needed to show non-equivalence, but please turn in the entire table so that we can give partial credit in case of errors.) Your truth table must include columns for all subformulas.

a) $P \leftrightarrow Q$ vs. $\neg P \leftrightarrow \neg Q$

b) $(P \rightarrow Q) \leftrightarrow P$ vs. $\neg Q$

c) $P \rightarrow Q$ vs. $(P \wedge R) \rightarrow (Q \wedge R)$

d) $(P \rightarrow Q) \rightarrow R$ vs. $P \rightarrow (Q \rightarrow R)$

e) $(P \wedge Q) \rightarrow R$ vs. $P \rightarrow (Q \rightarrow R)$

Task 2 – Tool Cool For Rule [20 pts]

Prove the following assertions using a sequence of logical equivalences such as Associativity, Biconditional, Commutativity, De Morgan, Distributivity, Domination, Double Negation, Idempotency, Identity, Law of Implication, Negation. (Our solutions, put together, use every one of those rules at least once.)

Hints: For equivalences where one side is much longer than the other, a good heuristic is to start with the longer side and try to apply the rules that will shorten it. In some cases, it may work better to work to shorten both sides to the same expression and then combine those two sequences into one.

a) $P \to (Q \to R) \equiv (P \wedge Q) \to R$

b) $(\neg P \to P) \to P \equiv \top$

c) $(P \to Q) \wedge R \equiv (R \to P) \to (Q \wedge R)$

d) $(P \to Q) \wedge (\neg P \to Q) \equiv Q$ (proof by "simple cases")

e) $((P \wedge Q) \to P) \wedge (P \to (P \wedge Q)) \equiv P \to Q$

Submit and check your answers to this question here:

`http://cozy.cs.washington.edu`

You can make as many attempts as needed to find a correct answer.

Just to be safe, we will include the problem for 0 points on Gradescope, and you can submit your answer there if you have trouble with cozy, but cozy is where we'd like you to submit your answers

Documentation is available on the Cozy homepage, at the the link labelled "Docs" at the top of the page.

Task 3 – A Love-Gate Relationship [16 pts]

Consider the following boolean function C:

p	q	r	s	$C(p,q,r,s)$
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	1
0	1	0	1	0
0	1	1	0	1
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	1
1	1	0	1	0
1	1	1	0	1
1	1	1	1	1

a) Write a Boolean algebra expression for C in sum-of-products form.

b) Use Boolean algebra identities to simplify your expression from (a) down to an expression that includes *only 4 gates* (each of which is either AND, OR, or NOT).

You should format your work like an equivalence chain with one expression per line and with the name of the identity applied to produce that line written next to it. However, since we are using Boolean algebra notation, which does not include unnecessary parentheses, you should not include lines that apply Associativity, Commutativity, or Identity.

Hint: It may be necessary to *temporarily* make your expression bigger in order to fully simplify it by the end. Our solution applies the "Idempotency" rule at one point to turn $pqrs'$ into $pqrs' + pqrs'$, giving us a second copy of the expression to use for later simplification. (If it were never necessary to do this — make the expression temporarily bigger — simplification would be a much easier problem! Sadly, simplification is a very hard problem in general.)

c) Write a truth table for your simplified expression from part (b) and confirm that it matches the one used to define C in Task 3. As always, be sure to include all subexpressions as their own columns.

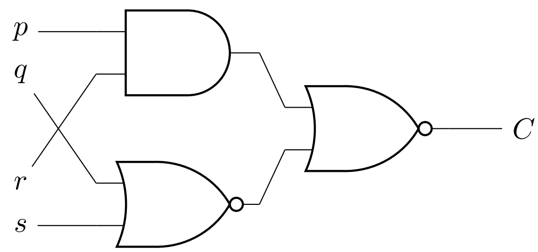
d) Draw your simplified expression from part (b) as a circuit.

Task 4 – Look What the SAT Dragged In [14 pts]

Suppose that we wish to find a circuit that calculates the following proposition, which we will call W:

$$(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r \wedge s) \vee$$
$$(\neg p \wedge q) \vee (\neg p \wedge \neg q \wedge s)$$

Our friend tells us that the circuit below always calculates the same value as W.



We would like to check that our friend is telling the truth. One way to do so would be to write out a truth table, but instead, we will see if we can use available software to do this for us. This problem guides you through how you could use a SAT solver (defined below) to answer this question.

a) First, translate our friend's circuit into a proposition using only $\neg$, $\wedge$, and $\vee$. Do not simplify.

b) Write a proposition that is *always true* iff the circuit above correctly calculates the value of W. (*Hint*: Use your answer to part (a).) Explain your answer. Again, do not simplify.

c) Explain why Task 1 part (a) tells us that $C \equiv D$ holds iff $\neg C \equiv \neg D$ holds.

Hint: one fact from lecture about equivalences will be useful here.

d) Explain why $A \equiv B$ holds iff $\neg(A \leftrightarrow B)$ is a *contradiction*.

Hint: it may be useful to part (c) with an appropriate choice for the expressions C and D. (That is not necessarily the only way to explain this, but that is how our solution does it.)

e) Write a proposition that is *always false* (i.e., a contradiction) iff the circuit above correctly calculates the value of W. (*Hint*: Refer back to part (b) and use (d).) Explain your answer. Do not simplify.

In the **Satisfiability** problem, we are given a proposition and asked whether there is *any* assignment of the variables that will make it true. In other words, a proposition is satisfiable iff its truth table has at least one row with value T.

Suppose we have access to software that solves the Satisfiability problem. Such a piece of software is referred to as a SAT solver. A SAT solver accepts a proposition A as input, and returns true iff A is satisfiable. Let's see how to use a SAT solver to finish checking our friend's work

f) Explain why any proposition A is satisfiable iff A is not a contradiction.

g) Explain how we can use a SAT solver to determine whether the circuit correctly calculates the value of W. Specifically, say what input we should pass to the SAT solver and how we can use its output to determine if the circuit is correct. (*Hint*: Use your answers to parts (e) and (f).)

Task 5 – A Guide to Proper Manners and Predicate [16 pts]

Let the domain of discourse be course staff and hobbies. Define the predicates Staff(x) to mean that x is a member of the course staff and Hobby(y) to mean that y is a hobby. Define the predicate Enjoys(x, y) to mean that x enjoys y (presumably x is a person and y is a hobby) and the predicate RequiresSkill(y) to mean that y requires skill (presumably y is a hobby).

Translate each of the following logical statements into English. You should not simplify. However, you should use the techniques shown in lecture for producing more natural translations when restricting domains and for avoiding the introduction of variable names when not necessary.

a) $\neg\exists y\,(\text{Hobby}(y) \wedge \text{RequiresSkill}(y))$

b) $\exists x\,\big(\text{Staff}(x) \wedge \exists y\,(\text{Hobby}(y) \wedge \text{Enjoys}(x, y))\big)$

c) $\exists x\,\big(\text{Staff}(x) \wedge \exists y\,(\text{Hobby}(y) \wedge \text{Enjoys}(x, y) \wedge \neg\,\text{RequiresSkill}(y))\big)$

d) $\forall y\,\big((\text{Hobby}(y) \wedge \text{RequiresSkill}(y)) \rightarrow \exists x\,(\text{Staff}(x) \wedge \text{Enjoys}(x, y))\big)$

Task 6 – Animal, Vegetable, or Literal [12 pts]

Let P and Q be predicates.

a) Translate the proposition

$$\forall x\,(P(x) \rightarrow Q(x))$$

directly into English. This time, *do not* try to make your translation natural sounding. Just do the most literal translation possible.

b) Translate the proposition

$$(\forall x\,P(x)) \rightarrow (\forall x\,Q(x))$$

directly into English. Again, *do not* try to make your translation natural sounding. Just do the most literal translation possible.

c) Give an example of predicates P and Q and a domain of discourse so that the propositions from parts a and b **do not** have the same truth value (i.e., one is false and one is true).

d) Give an example of predicates P and Q and a domain of discourse where the propositions from parts a and b **do** have the same truth value (i.e., both are false or both are true).

Task 7 – Extra Credit: Coined at the Hip

Five pirates, called Ann, Brenda, Carla, Danielle and Emily, found a treasure of 100 gold coins. On their ship, they decide to split the coins using the following scheme:

- The first pirate in alphabetical order becomes the chief pirate.
- The chief proposes how to share the coins, and all other pirates (except the chief) vote for or against it.
- If 50% or more of the pirates vote for it, then the coins will be shared that way.
- Otherwise, the chief will be thrown overboard, and the process is repeated with the pirates that remain.

Thus, in the first round Ann is the chief: if her proposal is rejected, she is thrown overboard and Brenda becomes the chief, etc; if Ann, Brenda, Carla, and Danielle are thrown overboard, then Emily becomes the chief and keeps the entire treasure.

The pirates' first priority is to stay alive: they will act in such a way as to avoid death. If they can stay alive, they want to get as many coins as possible. Finally, they are a blood-thirsty bunch, if a pirate would get the same number of coins if she voted for or against a proposal, she will vote against so that the pirate who proposed the plan will be thrown overboard.

Assuming that all 5 pirates are intelligent (and aware that all the other pirates are just as aware, intelligent, and bloodthirsty), what will happen? Your solution should indicate which pirates die, and how many coins each of the remaining pirates receives.