CSE 311: Foundations of Computing I

Set Theory

Well-Known Sets

- $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of *Natural Numbers*.
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is the set of *Integers*.
- $\mathbb{Q} = \{p/q \,:\, p, q \in \mathbb{Z} \land q \neq 0\}$ is the set of *Rational Numbers*.
- $\mathbb{R}$ is the set of *Real Numbers*.

Set in Logic

- Every set gives rise to a predicate "$x \in S$" that is true iff x is an element of the set.
- The shorthand "$x \notin S$" means $\neg(x \in S)$.
- Sets can be defined from predicates using "set builder" notation: $S \;::=\; \{x \,:\, P(x)\}$
- Inference rules for definitions now apply to all sets defined from predicates:

Def of S
$\dfrac{x \in S}{\therefore \quad P(x)}$

Undef S
$\dfrac{P(x)}{\therefore \quad x \in S}$

- The shorthand "$\forall x \in S\,(Q(x))$" means $\forall x\,((x \in S) \to Q(x))$.
  The shorthand "$\exists x \in S\,(Q(x))$" means $\exists x\,((x \in S) \land Q(x))$.

Set Operations

Let A, B be sets. We can define new sets from A and B:

- $A \cup B$ is the *union* of A and B: $A \cup B \;::=\; \{x \,:\, (x \in A) \lor (x \in B)\}$
- $A \cap B$ is the *intersection* of A and B: $A \cap B \;::=\; \{x \,:\, (x \in A) \land (x \in B)\}$
- $A \setminus B$ is the *difference* of A and B: $A \setminus B \;::=\; \{x \,:\, (x \in A) \land \neg(x \in B)\}$
- $A \oplus B$ is the *symmetric difference* of A and B: $A \oplus B \;::=\; \{x \,:\, (x \in A) \oplus (x \in B)\}$
- $\overline{A}$ is the *complement* of A with respect to "universe" $\mathcal{U}$: $\overline{A} \;::=\; \{x \,:\, (x \in \mathcal{U}) \land \neg(x \in A)\}$.[1]
- $A \times B$ is the *Cartesian product* of A and B: $A \times B \;::=\; \{x \,:\, \exists a \in A,\, \exists b \in B\,(x = (a, b))\}$
- $\mathcal{P}(A)$ is the *Power Set* of A, whose elements are themselves sets: $\mathcal{P}(A) \;::=\; \{B \,:\, B \subseteq A\}$

Set Comparison

Let A, B be sets. We can define new predicates that compare A and B:

- A *equals* B when they have the same elements: $A = B \;::=\; \forall x\,((x \in A) \leftrightarrow (x \in B))$
- A is a *subset* of B when B contains all of A's elements: $A \subseteq B \;::=\; \forall x\,((x \in A) \to (x \in B))$
- Theorem: $A = B$ iff $A \subseteq B$ and $B \subseteq A$

[1] If $\mathcal{U}$ is not specified, it is the entire domain of discourse.