Section 08: Induction, Regular Expressions, CFGs

1. Recursively Defined Sets of Strings

For each of the following, write a recursive definition of the sets satisfying the following properties. Briefly justify that your solution is correct.

(a) Binary strings of even length.

(b) Binary strings not containing 10.

(c) Binary strings not containing 10 as a substring and having at least as many 1s as 0s.

(d) Binary strings containing at most two 0s and at most two 1s.

2. Structural Induction

(a) Consider the following recursive definition of strings.

Basis Step: `""` is a string

Recursive Step: If X is a string and c is a character then $\mathsf{append}(c, X)$ is a string.

Recall the following recursive definition of the function len:

$$\begin{aligned} \mathsf{len}(\texttt{""}) &= 0 \\ \mathsf{len}(\mathsf{append}(c, X)) &= 1 + \mathsf{len}(X) \end{aligned}$$

Now, consider the following recursive definition:

$$\begin{aligned} \mathsf{double}(\texttt{""}) &= \texttt{""} \\ \mathsf{double}(\mathsf{append}(c, X)) &= \mathsf{append}(c, \mathsf{append}(c, \mathsf{double}(X))). \end{aligned}$$

Prove that for any string X, $\mathsf{len}(\mathsf{double}(X)) = 2\mathsf{len}(X)$.

(b) Consider the following definition of a (binary) **Tree**:

Basis Step: $\bullet$ is a **Tree**.

Recursive Step: If L is a **Tree** and R is a **Tree** then $\texttt{Tree}(\bullet, L, R)$ is a **Tree**.

The function leaves returns the number of leaves of a **Tree**. It is defined as follows:

$$\begin{aligned} \mathsf{leaves}(\bullet) &= 1 \\ \mathsf{leaves}(\texttt{Tree}(\bullet, L, R)) &= \mathsf{leaves}(L) + \mathsf{leaves}(R) \end{aligned}$$

Also, recall the definition of size on trees:

$$\begin{aligned} \mathsf{size}(\bullet) &= 1 \\ \mathsf{size}(\texttt{Tree}(\bullet, L, R)) &= 1 + \mathsf{size}(L) + \mathsf{size}(R) \end{aligned}$$

Prove that $\mathsf{leaves}(T) \geq \mathsf{size}(T)/2 + 1/2$ for all Trees T.

(c) Prove the previous claim using strong induction. Define $P(n)$ as "all trees T of size n satisfy $\textsf{leaves}(T) \geq \textsf{size}(T)/2 + 1/2$". You may use the following facts:

- For any tree T we have $\textsf{size}(T) \geq 1$.
- For any tree T, $\textsf{size}(T) = 1$ if and only if $T = \bullet$.

If we wanted to prove these claims, we could do so by structural induction.

Note, in the inductive step you should start by letting T be an arbitrary tree of size $k+1$.

3. Reversing a Binary Tree

Consider the following definition of a (binary) **Tree**.

Basis Step `Nil` is a **Tree**.

Recursive Step If L is a **Tree**, R is a **Tree**, and x is an integer, then $\texttt{Tree}(x, L, R)$ is a **Tree**.

The sum function returns the sum of all elements in a **Tree**.

$$\begin{aligned} \textsf{sum}(\texttt{Nil}) &= 0 \\ \textsf{sum}(\texttt{Tree}(x, L, R)) &= x + \textsf{sum}(L) + \textsf{sum}(R) \end{aligned}$$

The following recursively defined function produces the mirror image of a **Tree**.

$$\begin{aligned} \textsf{reverse}(\texttt{Nil}) &= \texttt{Nil} \\ \textsf{reverse}(\texttt{Tree}(x, L, R)) &= \texttt{Tree}(x, \textsf{reverse}(R), \textsf{reverse}(L)) \end{aligned}$$

Show that, for all **Tree**s T that

$$\textsf{sum}(T) = \textsf{sum}(\textsf{reverse}(T))$$

4. Walk the Dawgs

Suppose a dog walker takes care of $n \geq 12$ dogs. The dog walker is not a strong person, and will walk dogs in groups of 3 or 7 at a time (every dog gets walked exactly once). Prove the dog walker can always split the n dogs into groups of 3 or 7.

5. For All

For this problem, we'll see an incorrect use of induction. For this problem, we'll think of all of the following as binary trees:

- A single node.
- A root node, with a left child that is the root of a binary tree (and no right child)
- A root node, with a right child that is the root of a binary tree (and no left child)
- A root node, with both left and right children that are roots of binary trees.

Let $P(n)$ be "for all trees of height n, the tree has an odd number of nodes"

Take a moment to realize this claim is false.

Now let's see an incorrect proof:

We'll prove $P(n)$ for all $n \in \mathbb{N}$ by induction on n.

Base Case ($n = 0$): There is only one tree of height 0, a single node. It has one node, and $1 = 2 \cdot 0 + 1$, which is an odd number of nodes.

Inductive Hypothesis: Suppose $P(i)$ holds for $i = 0, ..., k$, for some arbitrary $k \geq 0$.

Inductive Step: Let T be an arbitrary tree of height k. All trees with nodes (and since $k \geq 0$, T has at least one node) have a leaf node. Add a left child and right child to a leaf (pick arbitrarily if there's more than one), This tree now has height $k+1$ (since T was height k and we added children below). By IH, T had an odd number of nodes, call it $2j+1$ for some integer j. Now we have added two more, so our new tree has $2j+1+2 = 2(j+1)+1$ nodes. Since j was an integer, so is $j+1$, and our new tree has an odd number of nodes, as required, so $P(k+1)$ holds.

By the principle of induction, $P(n)$ holds for all $n \in \mathbb{N}$. Since every tree has an (integer) height of 0 or more, every tree is included in some $P()$, so the claim holds for all trees.

(a) What is the bug in the proof?

(b) What should the starting point and target of the IS be (you can't write a full proof, as the claim is false).

6. Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

7. Induction with Formulas

These problems are a little more difficult and abstract. Try making sure you can do all the other problems before trying these ones.

(a) (i) Show that given two sets A and B that $\overline{A \cup B} = \overline{A} \cap \overline{B}$. (Don't use induction.)

(ii) Show using induction that for an integer $n \geq 2$, given n sets $A_1, A_2, \ldots, A_{n-1}, A_n$ that

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_{n-1} \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{n-1}} \cap \overline{A_n}$$

(b) (i) Show that given any integers a, b, and c, if $c \mid a$ and $c \mid b$, then $c \mid (a+b)$. (Don't use induction.)

(ii) Show using induction that for any integer $n \geq 2$, given n numbers $a_1, a_2, \ldots, a_{n-1}, a_n$, for any integer c such that $c \mid a_i$ for $i = 1, 2, \ldots, n$, that

$$c \mid (a_1 + a_2 + \cdots + a_{n-1} + a_n).$$

In other words, if a number divides each term in a sum then that number divides the sum.

8. One-to-One and Onto

For each of these functions, state whether it is one-to-one, onto, both, or neither.

(a) $f : \mathbb{N} \to \mathbb{N}$, $f(x) = x^2$

(b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

(c) $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = x^2$, where $\mathbb{R}^+ = \{x : x \in \mathbb{R} \land x \geq 0\}$, i.e., the set of non-negative real numbers.

Notice that the domain and co-domain matter! You have to know both to tell whether the function is one-to-one or onto.

9. A Bijection Proof

Let A be the set of negative integers, i.e., $A = \{-1, -2, -3, ...\}$; let B be the set of integers at least 10, i.e., $B = \{10, 11, 12, 13, ...\}$ Show that $f : A \to B$ defined by $f(x) = |x| + 9$ is a bijection.

You may use these facts:

- for negative numbers x, y: $|x| = |y| \to x = y$
- for negative numbers $|x| = -x$

that