Functions And Graphs

CSE 311 Autumn 2024
Lecture 20

Logistics

Midterm - Wednesday 11/13

No lecture – Monday 11/11 (Veterans Day)

Review Session – Monday 11/11, 5:30 PM in CSE2 G01

- Practicing with an old midterm (likely au23)

Today

Function definitions

Graph definitions

Functions

Some types of functions

Why?

We’ll want to talk about sizes of infinite sets during the last week of classes. It’ll help us find problems our computers can’t solve.

Ok, but why now?

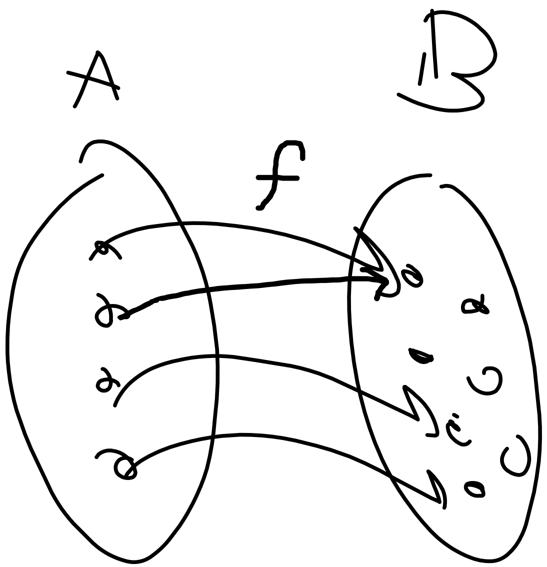
It’ll let us practice set proofs a bit more over the next few weeks!

Functions!

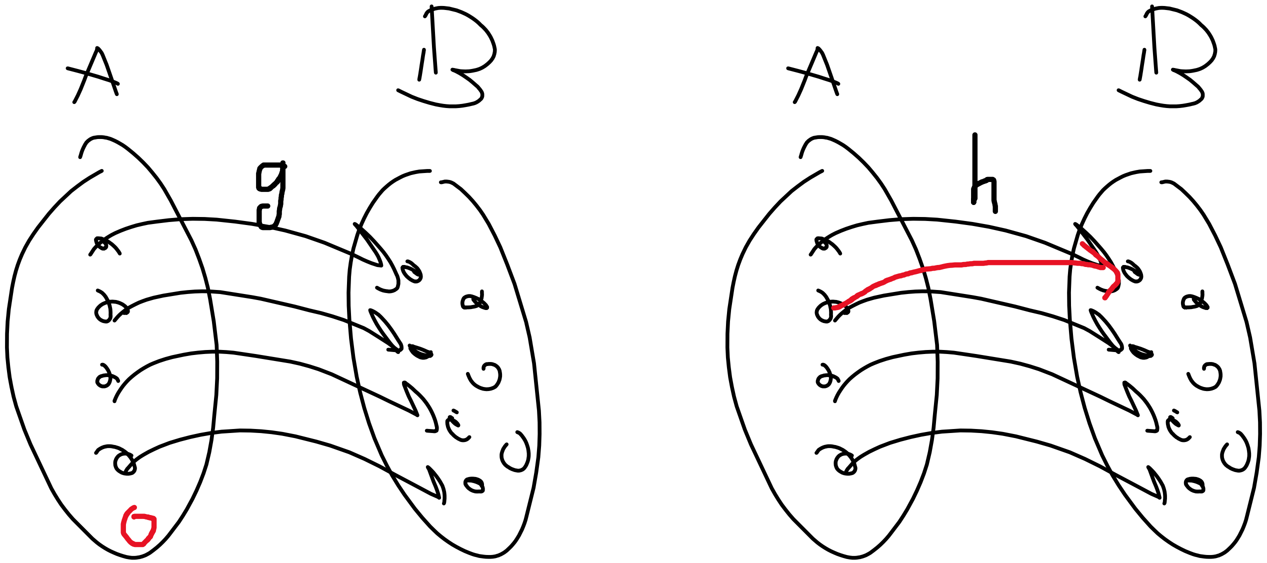
A function $f: A \to B$ maps every element of A to one element of B

A is the "domain", B is the "co-domain"

Good function



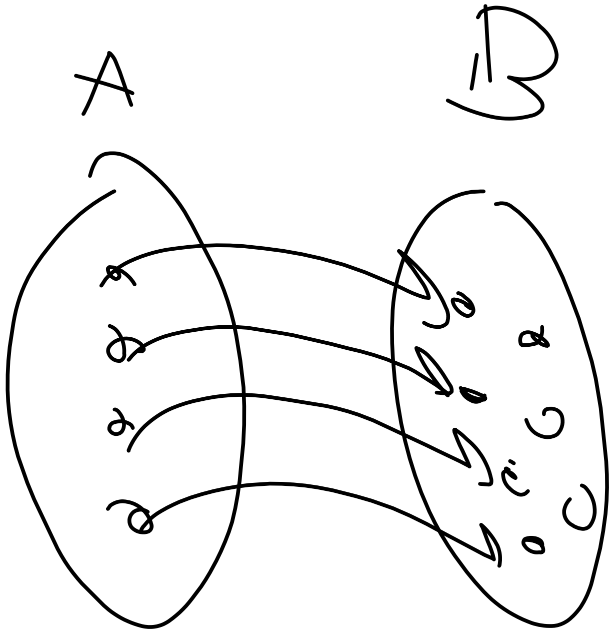
Not a function



Two Requirements for a Bijection

A function $f: A \to B$ maps every element of A to one element of B

A is the "domain", B is the "co-domain"



One-to-one (aka injection)

A function f is one-to-one iff
$\boldsymbol{\forall a \forall b (f(a) = f(b) \to a = b)}$

That is, every output has at most one possible input.

One-to-one (injection)

What did that definition say?

$\forall a \forall b (f(a) = f(b) \to a = b)$

In contrapositive that looks like

$\forall a \forall b \big(a \neq b \to f(a) \neq f(b)\big)$

So, if you get two different inputs, then you get two different outputs.

One-to-one proofs

One-to-one (aka injection)

A function f is one-to-one iff
$\forall a \forall b (f(a) = f(b) \to a = b)$

It's a for-all statement! We know how to prove it.

Let $f: \mathbb{Z} \to \mathbb{Z}$ be the function given by $f(x) = x + 5$.

Claim: f is one-to-one

Proof:

What's the outline? What do we introduce, what do we assume, what's our target?

One-to-one proofs

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Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be the function given by $f(x) = x + 5$.

Claim: f is one-to-one

Proof: Let a, b be arbitrary elements of our domain, and suppose $f(a) = f(b)$.

…

$a = b$

One-to-one proofs

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Claim: f is one-to-one

Proof: Let a, b be arbitrary elements of our domain, and suppose $f(a) = f(b)$.

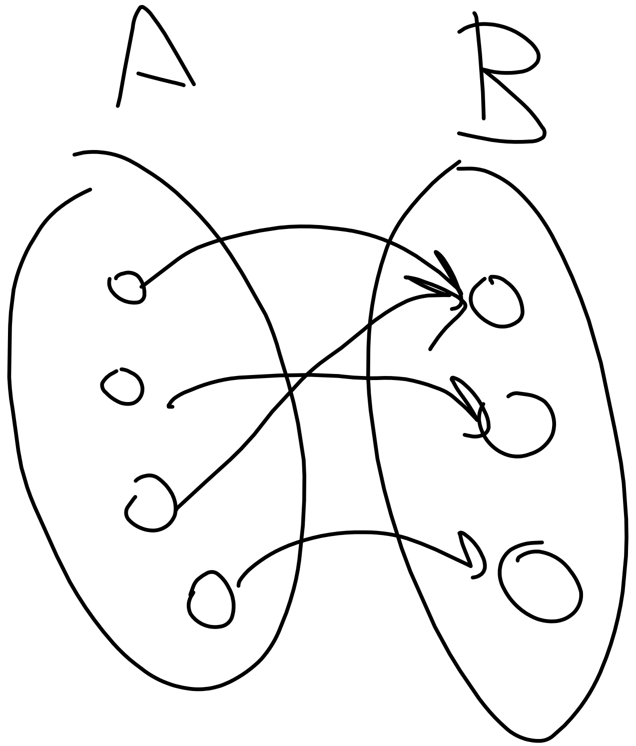
By definition of the function, we have $a + 5 = b + 5$

Subtracting 5 from each side, we have $a = b$, meeting the definition of one-to-one.

Two Requirements for a Bijection

A function $f: A \to B$ maps every element of A to one element of B

A is the "domain", B is the "co-domain"



Onto (aka surjection)

A function $f: A \to B$ is onto iff
$\forall b \in B \exists a \in A(b = f(a))$

Every output has at least one input that maps to it.

Onto proofs

Onto (aka surjection)

A function $f: A \rightarrow B$ is onto iff

$\forall b \in B \exists a \in A(b = f(a))$

It's a for-all statement, with an exists inside! We know how to prove it.

Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be the function given by $f(x) = x + 5$.

Claim: f is onto

Proof:

What's the outline? What do we introduce, what do we assume, what's our target?

Onto proofs

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Let $f: \mathbb{Z} \to \mathbb{Z}$ be the function given by $f(x) = x + 5$.

Claim: f is onto

Proof: Let b be an arbitrary element of the codomain.

Consider $a =$...

…

So $f(a) = b$

Onto proofs

Onto (aka surjection)

A function $f: A \to B$ is onto iff
$\forall b \in B \exists a \in A (b = f(a))$

It's a for-all statement, with an exists inside! We know how to prove it.

Let $f: \mathbb{Z} \to \mathbb{Z}$ be the function given by $f(x) = x + 5$.

Claim: f is onto

Proof: Let b be arbitrary element of the codomain.

Consider $a = b - 5$

Observe that $f(a) = a + 5 = b - 5 + 5 = b$.

Since $b \in \mathbb{Z}$, a is also an integer so it is in the domain. Thus f meets the definition of onto.

Bijection

One-to-one (aka injection)

A function f is one-to-one iff
$\forall a \forall b (f(a) = f(b) \rightarrow a = b)$

Onto (aka surjection)

A function $f: A \rightarrow B$ is onto iff
$\forall b \in B \exists a \in A (b = f(a))$

Bijection

A function $f: A \rightarrow B$ is a bijection iff
f is one-to-one and onto

A bijection maps every element of the domain to **exactly** one element of the co-domain, and every element of the codomain to **exactly** one element of the domain.

Sizes of sets

How do we know two sets are the same size?

Easy. Count the number of elements in both.

That works great for finite sets, but ∞ isn't really a number we get to count to…

More Practical

What does it mean that two sets have the same size?

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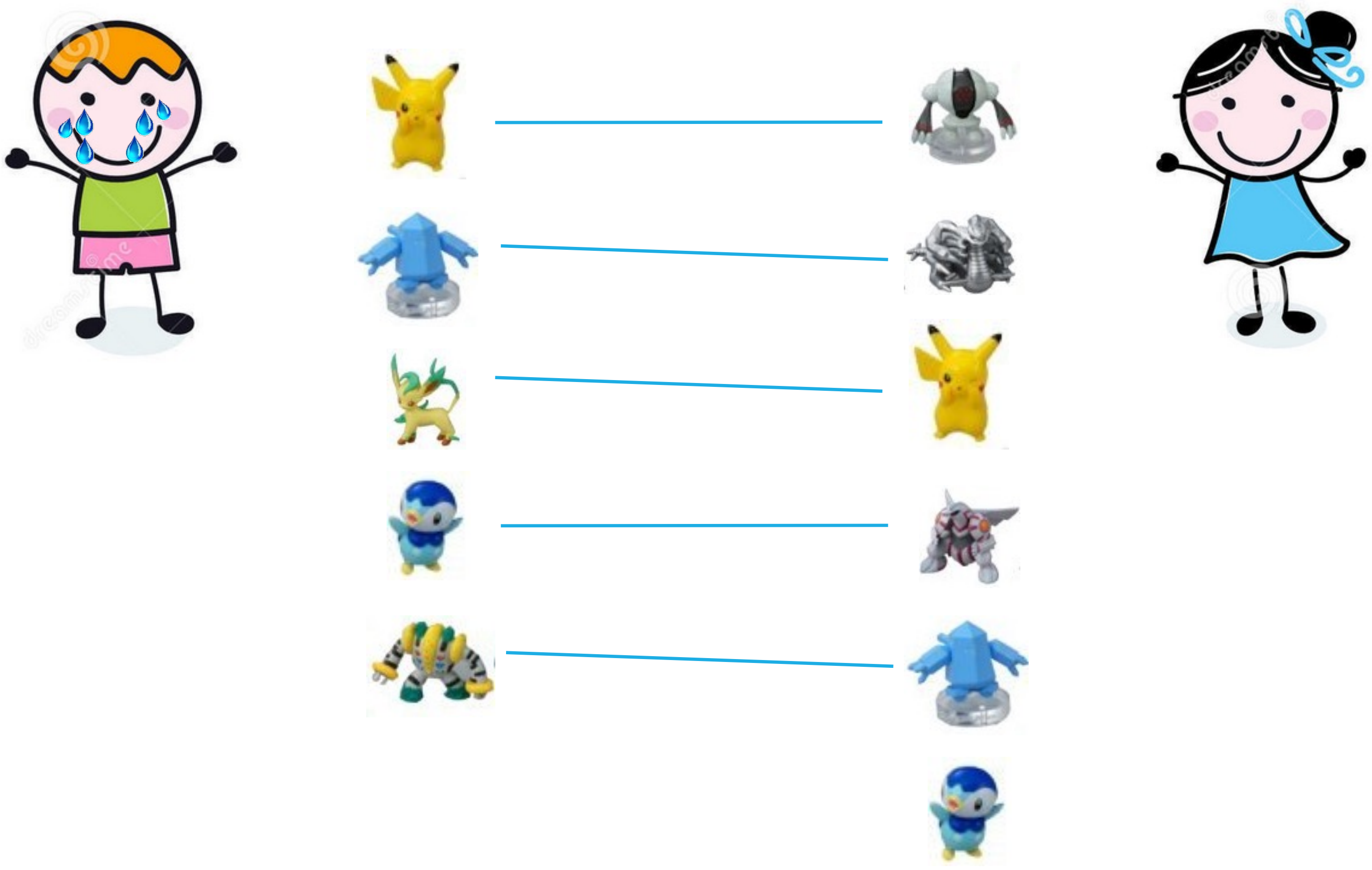


Why do we care about bijections?

Bijections create a (confusingly-named) one-to-one correspondence between sets.

There is a bijection $f: A \to B$ if and only if A and B are the same size.

A bijection “matches the elements up”

For finite sets we usually tell which of two sets is bigger by counting the number of elements in each and comparing the numbers.

These functions let you compare set sizes even if you can’t count the elements. We’ll use that idea for infinite sets in a few weeks.

Definition

Two sets A, B have the same size (same cardinality) if and only if there is a bijection $f: A \to B$

This matches our intuition on finite sets.

But it also works for infinite sets!

Let's see just how infinite these sets are.

Some infinite sets

Two sets A, B have the same size (same cardinality) if and only if there is a bijection $f: A \to B$

Let's compare the sizes of: $\mathbb{N}$, $\mathbb{Z}$, $\{x : x \text{ is an even integer}\}$

Some infinite sets

Two sets A, B have the same size (same cardinality) if and only if there is a bijection $f: A \to B$

Let's compare the sizes of: $\mathbb{N}$, $\mathbb{Z}$, $\{x : x \text{ is an even integer}\}$

$\mathbb{N}$ 0 1 2 3 4 5 6 7 $\ldots$

$\mathbb{Z}$

Some infinite sets

Two sets A, B have the same size (same cardinality) if and only if there is a bijection $f: A \to B$

Let's compare the sizes of: $\mathbb{N}$, $\mathbb{Z}$, $\{x : x$ is an even integer$\}$

$$\mathbb{Z} \quad \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

$$Even$$

They're all the same size.

$\mathbb{Z}$ and even integers?

$f(x) = 2x$ Is it a bijection?

One-to-one? Let $a, b \in \mathbb{Z}$ be arbitrary. Suppose $f(a) = f(b)$. By definition of f, $2a = 2b$. Dividing by 2, $a = b$.

Onto? Let b be an arbitrary even integer. Since b is even, there must be some a $\in \mathbb{Z}$ such that $b = 2a$. By definition of f, $f(a) = b$.

They're all the same size.

$\mathbb{Z}$ and even integers?

$f(x) = 2x$ Is it a bijection?

YES

$\mathbb{N}$ and $\mathbb{Z}$?

They're all the same size.

$\mathbb{Z}$ and even integers?

$f(x) = 2x$ Is it a bijection?

YES

$\mathbb{N}$ and $\mathbb{Z}$

$$g(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ -\frac{x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$

They're all the same size...

$\mathbb{N}$ and even integers?

$f(g(x))$ will work nicely. You can also build one explicitly.

Good exercise: show that if f and g are bijections then $f \circ g$ is also a bijection.

They're all the same size…

$\mathbb{N}$ and even integers?

$f(g(x))$ will work nicely. You can also build one explicitly.

$$\mathbb{N} \xrightarrow{g} \mathbb{Z} \xrightarrow{f} \text{evens}$$

Good exercise: show that if f and g are bijections then $f \circ g$ is also a bijection.

Countable

Countable

The set A is countable iff there's a one-to-one function from $\boldsymbol{A}$ to $\mathbb{N}$, Equivalently, $\boldsymbol{A}$ is countable iff it is finite or there is a bijection from $\boldsymbol{A}$ to $\mathbb{N}$

$\mathbb{N}, \mathbb{Z}, \{x : x$ is an even integer$\}$ are all countable.

To build a bijection from $\boldsymbol{A}$ to $\mathbb{N}$, just list all the elements!

Let's Try one that's a little harder

What about $\mathbb{Q}$. There's gotta be more of those right?

It's pretty intuitive to think there are more rationals than integers.

Between every two rationals, there's another rational number.

Or said in more intimidating fashion: between every two rationals there are infinitely many others!

The set of positive rational numbers

1/1	**1/2**	**1/3**	**1/4**	**1/5**	**1/6**	**1/7**	**1/8**	**…**
2/1	**2/2**	**2/3**	**2/4**	**2/5**	**2/6**	**2/7**	**2/8**	**…**
3/1	**3/2**	**3/3**	**3/4**	**3/5**	**3/6**	**3/7**	**3/8**	**…**
4/1	**4/2**	**4/3**	**4/4**	**4/5**	**4/6**	**4/7**	**4/8**	**…**
5/1	**5/2**	**5/3**	**5/4**	**5/5**	**5/6**	**5/7**	**…**	
6/1	**6/2**	**6/3**	**6/4**	**6/5**	**6/6**	**…**		
7/1	**7/2**	**7/3**	**7/4**	**7/5**	**….**			
…	**…**	**…**	**…**	**…**				

In bijection with the natural numbers

Order the rationals by their denominator (increasing), breaking ties by numerator.

$1/1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 1/6, \ldots$

$f(x) =$the x^{th} number in that list (indexed from 0)

That's a bijection from $\mathbb{N}$ to $\mathbb{Q}^+$(it's not a nice clean formula, but it's definitely a function)

Are all infinite sets countable?

No. We will prove this in a few weeks.

$\mathbb{R}$ **is uncountable**

Graphs

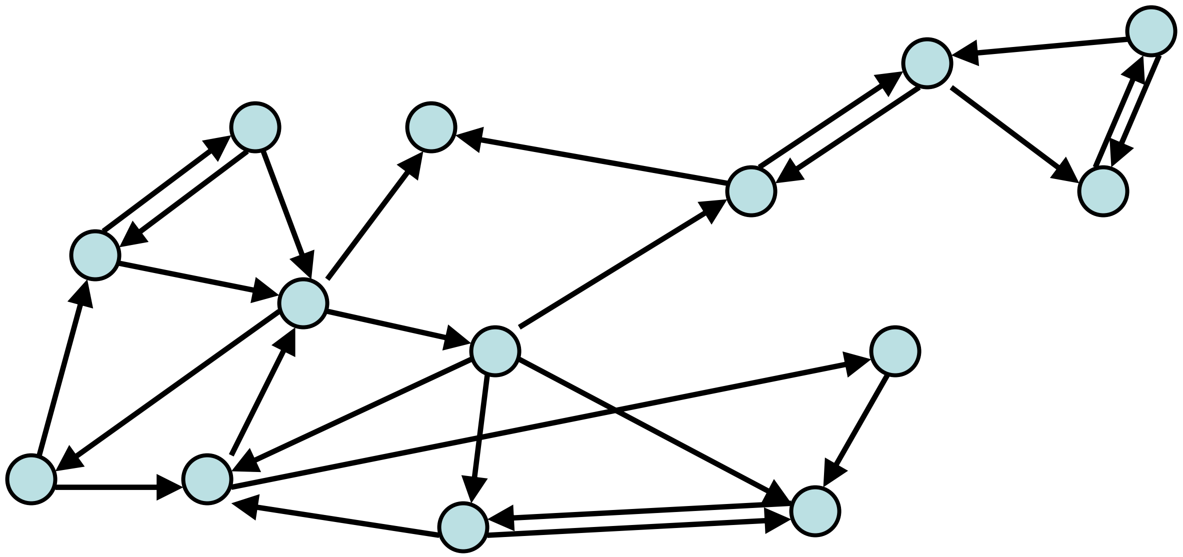
Directed Graphs

$G = (V, E)$

V is a set of vertices (an underlying set of elements)

E is a set of edges (ordered pairs of vertices; i.e. connections from one to the next).

Path $v_0, v_1, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$
Simple Path: path with all v_i distinct
Cycle: path with $v_0 = v_k$ (and $k > 0$)
Simple Cycle: simple path plus edge (v_k, v_0) with $k > 0$



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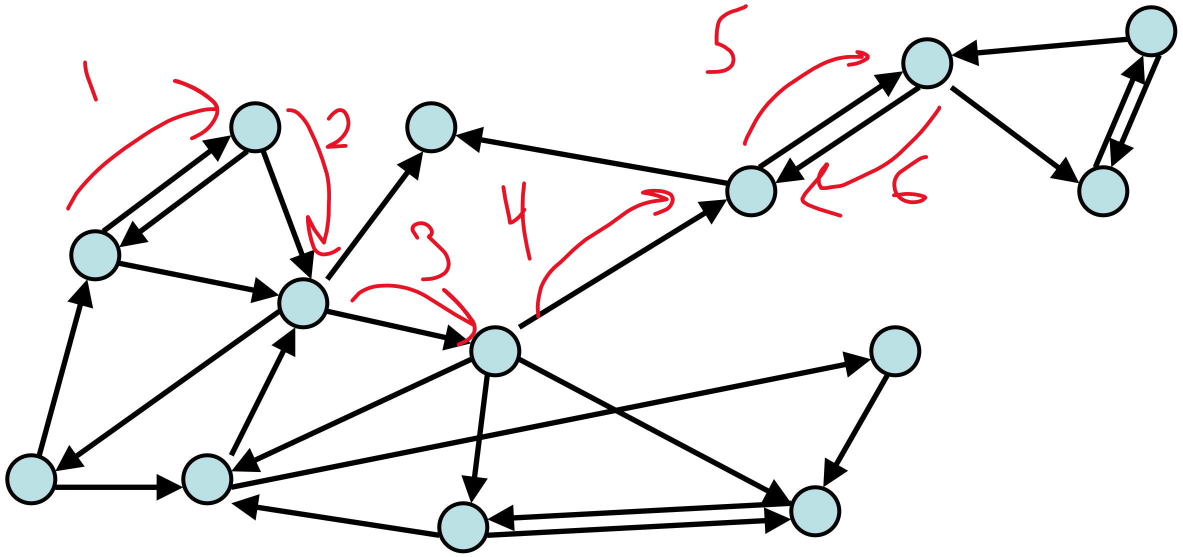

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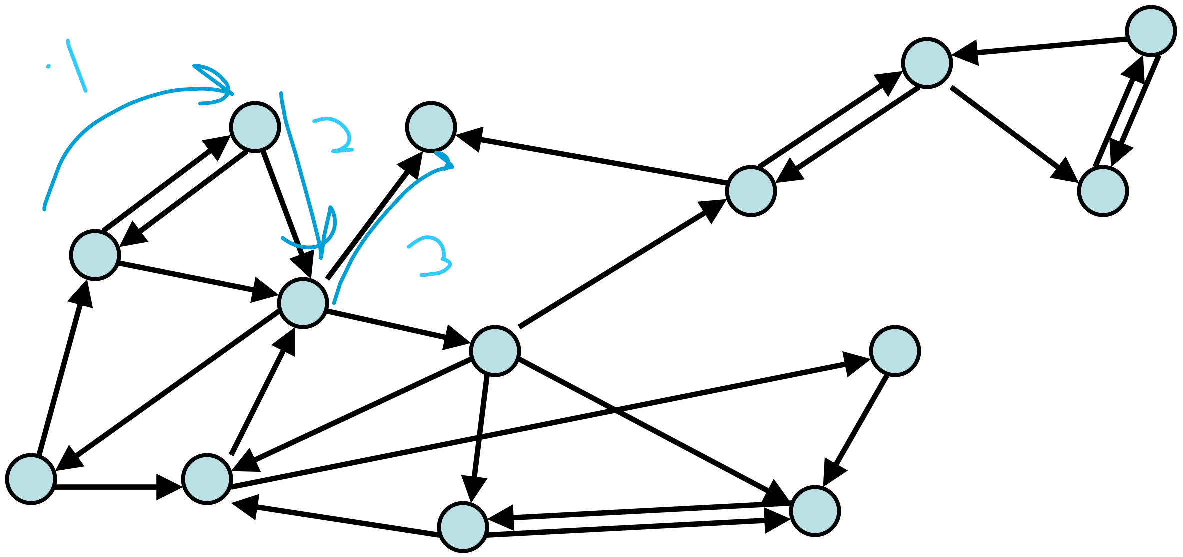

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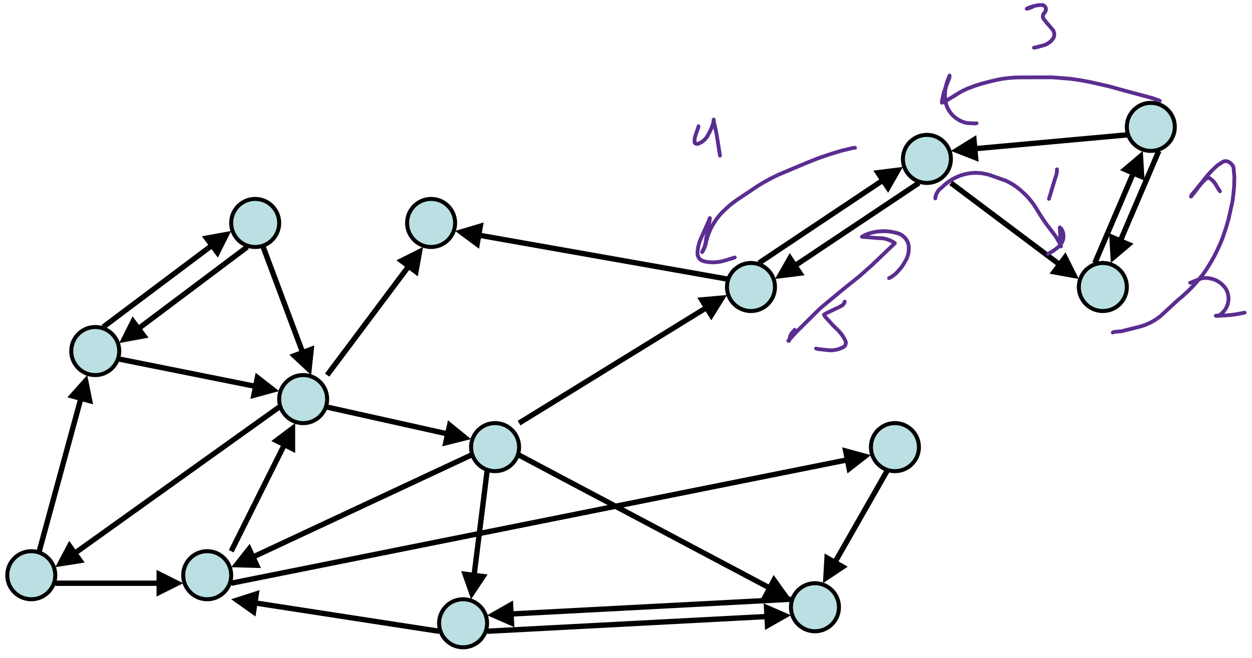

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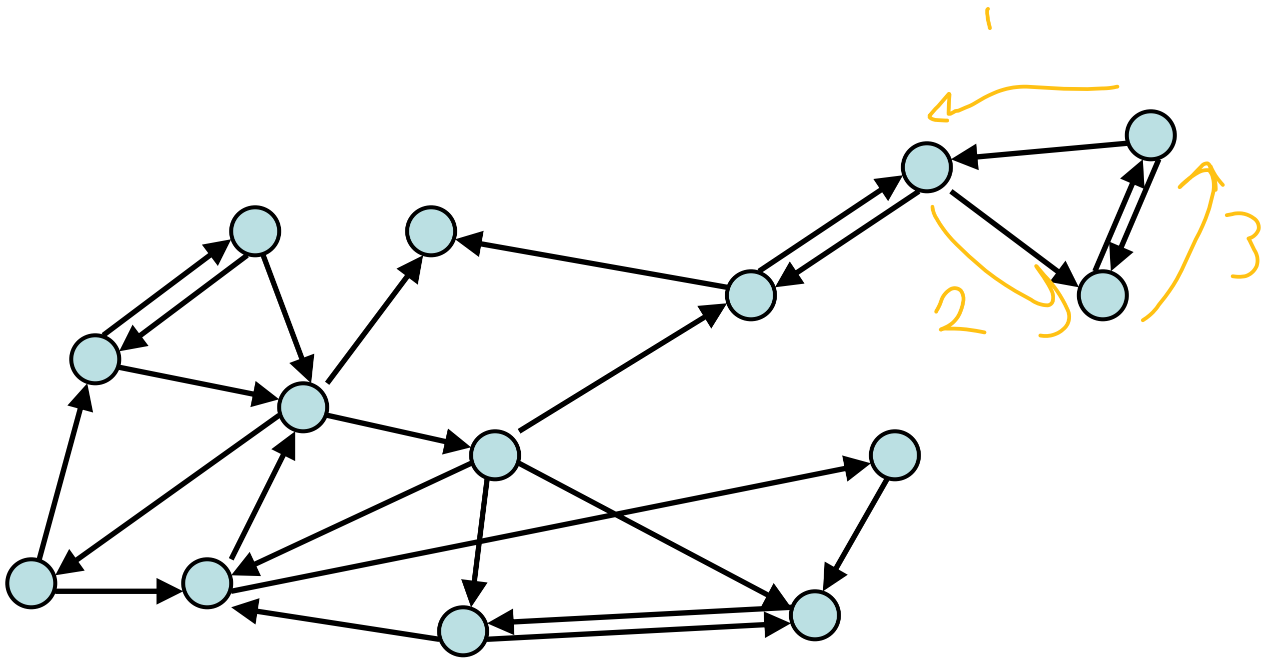

Extra examples

The 2nd and 3rd problems might be good review for the midterm ☺

Draw the graph!

Let $G = (V, E)$.

V = {1, 2, 3, 4}

E = {(1, 1), (1, 2), (2, 3), (3, 4), (4, 1)}

Is there a cycle?

Is there a simple cycle?

Draw the graph!

Let $G = (V, E)$.

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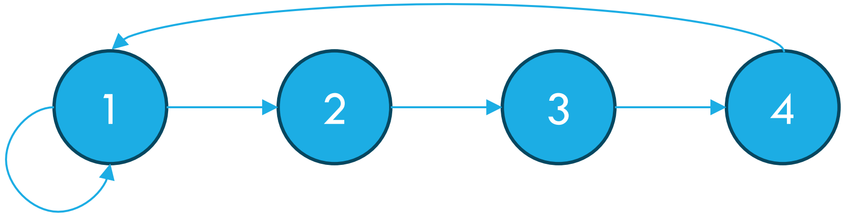
E = {(1, 1), (1, 2), (2, 3), (3, 4), (4, 1)}

Is there a cycle?

Is there a simple cycle?

Yes to both! Consider 1, 2, 3, 4, 1.



Between every two distinct rationals, there's another rational number.

Let $a, b \in \mathbb{Q}$ be arbitrary and suppose $a \neq b$. By the definition of rational, there's some integers p, r and non-zero integers q, s such that $a = \frac{p}{q},\ b = \frac{r}{s}$. Without loss of generality, assume $a < b$.

Consider $c = \frac{a+b}{2}$ (the average of a, b).

Since $a < b, a + b < 2b$ holds and we have $\mathrm{c} = \frac{a+b}{2} < b$. Similarly, $a < b$ gives $2a < a + b$ and we have $a < \frac{a+b}{2} = c$. Thus, c is "between" a, b.

$c = \frac{a+b}{2} = \frac{\frac{p}{q}+\frac{r}{s}}{2} = \frac{ps+rq}{2qs}$. As p, q, r, s are integers and q, s non-zero, $ps + rq$ is an integer and $2qs$ is a non-zero integer. So c is rational by definition of rational.

So c is a rational number between a, b. Since a, b were arbitrary, we can find a rational number between any two distinct rational numbers.

There are infinitely many rational numbers between any two distinct rational numbers.

Let $a, b \in \mathbb{Q}$ be arbitrary and suppose $a \neq b$.

For the sake of contradiction, suppose there are finitely many rational numbers between a, b. Then we can list all of them (rational numbers have ordering so we can list them from least to greatest):

$$a = p_1 < \cdots < p_n = b \text{ where each } p_i \in \mathbb{Q}$$

Notice that p_1, p_2 are distinct rational numbers. By the previous proof, there's a rational number q between these 2 distinct rational numbers. But $a < q < b$ and q isn't in our list, so we have a contradiction.

Since a, b were arbitrary, between 2 distinct rationals there are infinitely many rationals!

Composition of 2 bijections is a bijection.

Let $f: B \to C, g: A \to B$ be arbitrary bijections.

Consider the composition $f \circ g\colon\ A \to C$. We will use the alternative notation $f\big(g(x)\big)$ for clarity.

(1) We show $f\big(g(x)\big)$ is **one-to-one**. Let $a, b \in A$ be arbitrary. Suppose $f\big(g(a)\big) = f(g(b))$. Since f is a bijection, it’s also one-to-one so $g(a) = g(b)$. Since g is a bijection, it’s also one-to-one so $a = b$. Since a, b were arbitrary, $f\big(g(x)\big)$ is one-to-one.

…

Thus, $f \circ g$ is a bijection because it’s both one-to-one and onto.

Composition of 2 bijections is a bijection.

Let $f: B \to C, g: A \to B$ be arbitrary bijections.

Consider the composition $f \circ g$: $A \to C$. We will use the alternative notation $f\big(g(x)\big)$ for clarity.

…

(2) We show $f\big(g(x)\big)$ is onto. Let $c \in C$ be arbitrary. Since f is onto, there is some $b \in B$ such that $f(b) = c$. Since g is onto, there is some $a \in A$ such that $g(a) = b$. Combining the facts that $f(b) = c$ and $g(a) = b$, we have $f(g(a)) = c$. Since c was arbitrary, $f\big(g(x)\big)$ is onto.

Thus, $f \circ g$ is a bijection because it's both one-to-one and onto.

Show that the following function is a bijection.

$$g(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ -\frac{x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$

(1) Let $a, b \in \mathbb{N}$ be arbitrary. Suppose that $g(a) = g(b)$. We go by cases:

$g(a) \geq 0$: So a, b even because $a, b \geq 0$

(non-negative outputs have even inputs, since $\frac{a}{2}, \frac{b}{2} > 0$):

$\frac{a}{2} = \frac{b}{2} \Rightarrow a = b$

$g(a) < 0$: So a, b odd because $a, b \geq 0$

(negative outputs have odd inputs, since $\frac{a+1}{2}, \frac{b+1}{2} > 0$):

$-\frac{a+1}{2} = -\frac{b+1}{2} \Rightarrow -a - 1 = -b - 1 \Rightarrow a = b$

These cases are exhaustive so $a = b$. Since $a, b \in \mathbb{N}$ were arbitrary, the function one-to-one.

Show that the following function is a bijection.

$$g(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ -\frac{x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$

(2) Let $b \in \mathbb{Z}$ be arbitrary. We go by cases:

$b < 0$: Consider $a = 2(-b) - 1$. $a > 0$ because $-b > 0$ and multiplying and adding positive integers results in a positive integer. So $a \in \mathbb{N}$ (the domain). a is odd, so rearranging with algebra (and using the appropriate definition for g in the last step):

$$a = -2b - 1 \Rightarrow a + 1 = -2b \Rightarrow -\frac{(a+1)}{2} = b \Rightarrow g(a) = b$$

$b \geq 0$: Left as an exercise to the reader.

Since b was arbitrary, the function is onto.

Since the function is one-to-one and onto, it's a bijection.