

Warm up:

What is the following recursively-defined set?

Basis Step: $4 \in S, 5 \in S$

Recursive Step: If $x \in S$ and $y \in S$ then $x - y \in S$

(Just)

More Structural Induction ~~and Regular Expressions~~

CSE 311 Autumn 2024
Lecture 19

A few logistics

We're starting to schedule the alternate exam; you'll hear from us this week if you've filled out the form.

Section tomorrow is going to be midterm review

We're also trying to schedule a separate review session, announcement coming when the logistics are set.

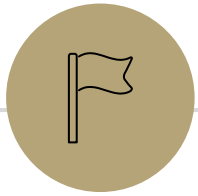
No lecture on Monday (Veteran's Day)

Most (maybe all?) Monday OH will be cancelled or moved.

CC20 (releases Friday) due next Friday (Nov 15)

No HW released tonight! We want you to study instead.

HW7 will come out after the midterm.



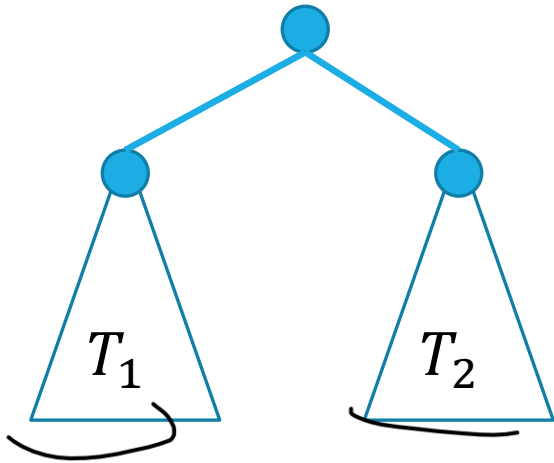
Trees!



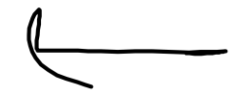
More Structural Sets

Binary Trees are another common source of structural induction.

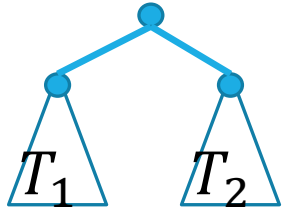
- ⌈ Basis: A single node is a rooted binary tree. ●
- ⌈ Recursive Step: If T_1 and T_2 are rooted binary trees with roots r_1 and r_2 , then a tree rooted at a new node, with children r_1, r_2 is a binary tree.




Functions on Binary Trees

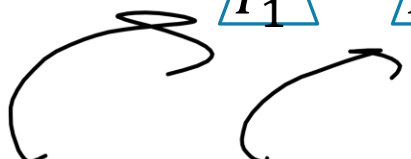
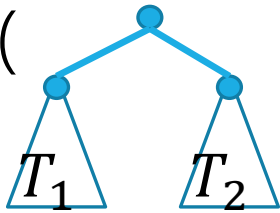
$\text{size}(\bullet) = 1$ 

$\text{size}(\text{tree}) = \text{size}(T_1) + \text{size}(T_2) + 1$




$\text{height}(\bullet) = 0$ 

$\text{height}(\text{tree}) = 1 + \max(\text{height}(T_1), \text{height}(T_2))$



Claim

We want to show that trees of a certain height can't have too many nodes. Specifically our claim is this:

$$\text{For all trees } T, \text{size}(T) \leq 2^{\text{height}(T)+1} - 1$$


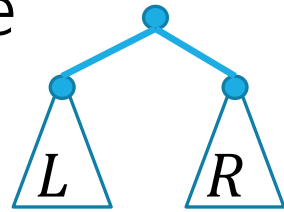
Take a moment to absorb this formula, then we'll do induction!

Structural Induction on Binary Trees

Let $P(T)$ be " $\text{size}(T) \leq 2^{\text{height}(T)+1} - 1$ ". We show $P(T)$ for all binary trees T by structural induction.

Base Case: Let $T = \bullet$. $\text{size}(T)=1$ and $\text{height}(T) = 0$, so $\text{size}(T)=1 \leq 2 - 1 = 2^{0+1} - 1 = 2^{\text{height}(T)+1} - 1$.

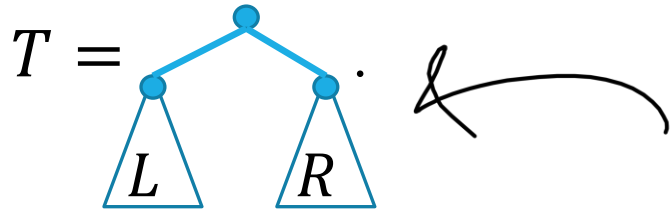
Inductive Hypothesis: Suppose $P(L)$ and $P(R)$ hold for arbitrary trees L, R . Let T be the tree



Inductive step: Figure out, (1) what we must show (2) a formula for height and a formula for size of T .

Structural Induction on Binary Trees (cont.)

Let $P(T)$ be " $\text{size}(T) \leq 2^{\text{height}(T)+1} - 1$ ". We show $P(T)$ for all binary trees T by structural induction.



$$\text{height}(T) = 1 + \max\{\text{height}(L), \text{height}(R)\}$$

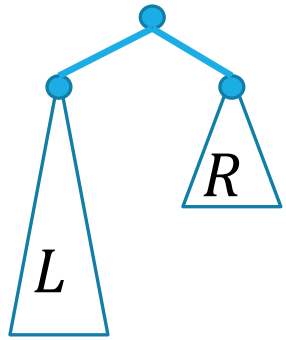
$$\text{size}(T) = 1 + \text{size}(L) + \text{size}(R)$$

$$\leq 1 + 2^{\text{height}(L)+1} - 1 + 2^{\text{height}(R)+1} - 1 \quad (\text{by I.A.})$$

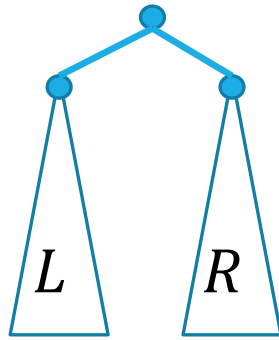
So $P(T)$ holds, and we have $P(T)$ for all binary trees T by the principle of induction.

How do heights compare?

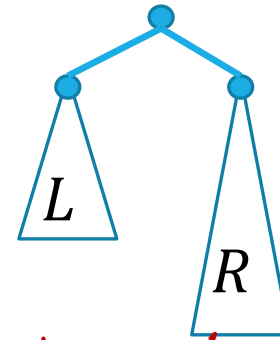
If L is taller than R ?



If L, R same height?



If R is taller than L ?



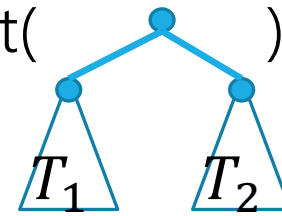
$$h(L) + 1 = h(T)$$

$$h(R) + 1 \leq h(T)$$

$$h(R) + 1$$

$$\text{height}(\bullet) = 0$$

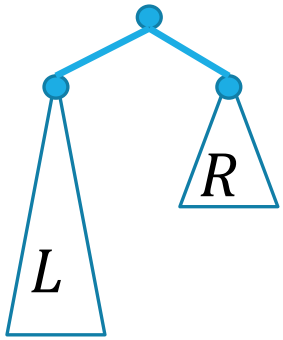
$$\text{height}(\text{tree}) =$$



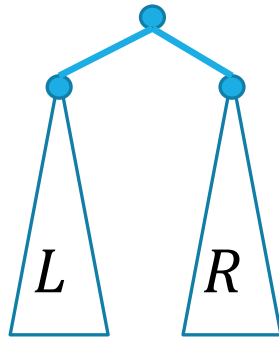
$$1 + \max(\text{height}(T_1), \text{height}(T_2))$$

How do heights compare?

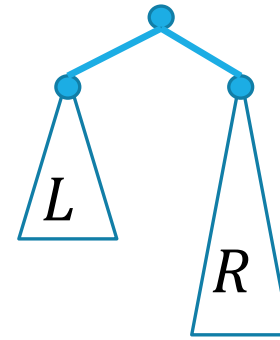
If L is taller than R ?



If L, R same height?



If R is taller than L ?



$$\text{height}(T) = \text{height}(L) + 1$$

$$\text{height}(T) > \text{height}(R) + 1$$

$$\text{height}(T) = \text{height}(L) + 1$$

$$\text{height}(T) = \text{height}(R) + 1$$

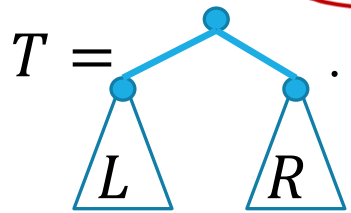
$$\text{height}(T) > \text{height}(L) + 1$$

$$\text{height}(T) = \text{height}(R) + 1$$

In all cases: $\text{height}(T) \geq \text{height}(L) + 1$, $\text{height}(T) \geq \text{height}(R) + 1$

Structural Induction on Binary Trees (cont.)

Let $P(T)$ be " $\text{size}(T) \leq 2^{\text{height}(T)+1} - 1$ ". We show $P(T)$ for all binary trees T by structural induction.



$$P(L): \text{size}(L) \leq 2^{\text{height}(L)+1} - 1$$

$$\text{height}(T) = 1 + \max\{\text{height}(L), \text{height}(R)\}$$

$$\text{size}(T) = 1 + \text{size}(L) + \text{size}(R)$$

$$\text{size}(T) = 1 + \text{size}(L) + \text{size}(R) \leq 1 + 2^{\text{height}(L)+1} - 1 + 2^{\text{height}(R)+1} - 1 \text{ (by IH)}$$

$$\leq 2^{\text{height}(L)+1} + 2^{\text{height}(R)+1} - 1 \text{ (cancel 1's)}$$

$$\leq 2^{\text{height}(T)} + 2^{\text{height}(T)} - 1 = 2^{\text{height}(T)+1} - 1 \text{ (T taller than subtrees)}$$

So $P(T)$ holds, and we have $P(T)$ for all binary trees T by the principle of induction.

Structural Induction Template

1. Define $P()$ State that you will show $P(x)$ holds for all $x \in S$ and that your proof is by structural induction.
2. Base Case: Show $P(b)$
[Do that for every b in the basis step of defining S]
3. Inductive Hypothesis: Suppose $P(x)$
[Do that for every x listed as already in S in the recursive rules].
4. Inductive Step: Show $P()$ holds for the "new elements."
[You will need a separate step for every element created by the recursive rules].
5. Therefore $P(x)$ holds for all $x \in S$ by the principle of induction.



Structural Induction on Strings

Strings

ε is "the empty string"

The string with 0 characters – `""` in Java (not `null`!)

Σ^* :

Basis: $\varepsilon \in \Sigma^*$.

Recursive: If $w \in \Sigma^*$ and $a \in \Sigma$ then $wa \in \Sigma^*$

wa means the string of w with the character a appended.

You'll also see $w \cdot a$ ($a \cdot$ to mean "concatenate" i.e. `+` in Java)

Functions on Strings

Since strings are defined recursively, most functions on strings are as well.

Length:

$$\text{len}(\varepsilon) = 0;$$

$$\text{len}(wa) = \text{len}(w) + 1 \text{ for } w \in \Sigma^*, a \in \Sigma$$

Reversal:

$$\varepsilon^R = \varepsilon;$$

$$(wa)^R = aw^R \text{ for } w \in \Sigma^*, a \in \Sigma$$

Concatenation

$$x \cdot \varepsilon = x \text{ for all } x \in \Sigma^*;$$

$$x \cdot (wa) = (x \cdot w)a \text{ for } w \in \Sigma^*, a \in \Sigma$$

Number of c 's in a string

$$\#_c(\varepsilon) = 0$$

$$\#_c(wc) = \#_c(w) + 1 \text{ for } w \in \Sigma^*;$$

$$\#_c(wa) = \#_c(w) \text{ for } w \in \Sigma^*, a \in \Sigma \setminus \{c\}.$$

Claim for all $x, y \in \Sigma^*$ $\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$.

Let $P(y)$ be "for all $x \in \Sigma^*$ $\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$."

Notice the strangeness of this $P()$ there is a "for all x " inside the definition of $P(y)$.

That means we'll have to introduce an arbitrary x as part of the base case and the inductive step!

Claim for all $x, y \in \Sigma^*$ $\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$.

Let $P(y)$ be " $\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$ for all $x \in \Sigma^*$."

We ~~prove~~ $P(y)$ for all $x \in \Sigma^*$ by structural induction.

Base Case:

Inductive Hypothesis

Inductive Step:

We conclude that $P(y)$ holds for all string y by the principle of induction. Unwrapping the definition of P , we get $\forall x \forall y \in \Sigma^* \text{len}(xy) = \text{len}(x) + \text{len}(y)$, as required.

Claim for all $x, y \in \Sigma^*$ $\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$.

Let $P(y)$ be " $\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$ for all $x \in \Sigma^*$."

We prove $P(y)$ for all $x \in \Sigma^*$ by structural induction.

Base Case: Let x be an arbitrary string, $\text{len}(x \cdot \epsilon) = \text{len}(x)$
 $= \text{len}(x) + 0 = \text{len}(x) + \text{len}(\epsilon)$

Inductive Hypothesis: Suppose $P(w)$ for an arbitrary string w .

Inductive Step:

We conclude that $P(y)$ holds for all string y by the principle of induction.
Unwrapping the definition of P , we get $\forall x \forall y \in \Sigma^* \text{len}(xy) = \text{len}(x) + \text{len}(y)$, as required.

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We prove $P(y)$ for all $x \in \Sigma^*$ by structural induction.

Base Case: Let x be an arbitrary string, $\text{len}(x \cdot \epsilon) = \text{len}(x)$
 $= \text{len}(x) + 0 = \text{len}(x) + \text{len}(\epsilon)$

Inductive Hypothesis: Suppose $P(w)$ for an arbitrary string w .

Inductive Step: Let $y = wa$ for an arbitrary $a \in \Sigma$. We show $P(y)$. Let x be an arbitrary string.

...

Therefore, $\text{len}(xy) = \text{len}(x) + \text{len}(y)$, as required.

We conclude that $P(y)$ holds for all string y by the principle of induction.
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Claim for all $x, y \in \Sigma^*$ $\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$.

Let $P(y)$ be " $\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$ for all $x \in \Sigma^*$."

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Base Case: Let x be an arbitrary string, $\text{len}(x \cdot \epsilon) = \text{len}(x) = \text{len}(x) + 0 = \text{len}(x) + \text{len}(\epsilon)$

Inductive Hypothesis: Suppose $P(w)$ for an arbitrary string w .

Inductive Step: Let $y = wa$ for an arbitrary $a \in \Sigma$. We show $P(y)$. Let x be an arbitrary string.

$\text{len}(xy) = \text{len}(xwa) = \text{len}(xw) + 1$ (by definition of len)

$= \text{len}(x) + \text{len}(w) + 1$ (by IH)

$= \text{len}(x) + \text{len}(wa)$ (by definition of len)

Therefore, $\text{len}(xy) = \text{len}(x) + \text{len}(y)$, as required.

We conclude that $P(y)$ holds for all string y by the principle of induction. Unwrapping the definition of P , we get $\forall x \forall y \in \Sigma^* \text{len}(xy) = \text{len}(x) + \text{len}(y)$, as required.

$$\forall x \text{len}(x \cdot w) = \text{len}(x) + \text{len}(w)$$

Why all those arbitraries?

Let $P(y)$ be " $\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$ for all $x \in \Sigma^*$."

$P(\varepsilon)$ is a for-all statement, introduce arbitrary variable to show for-all.

We prove $P(y)$ for all $x \in \Sigma^*$ by structural induction.

Base Case: Let x be an arbitrary string, $\text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + 0 = \text{len}(x) + \text{len}(\varepsilon)$

Needs to be arbitrary because it's in the IH (induction wouldn't show "all strings" otherwise)

Inductive Hypothesis: Suppose $P(w)$ for an arbitrary string w .

Inductive Step: Let $y = wa$ for an arbitrary $a \in \Sigma$. We show $P(y)$. Let x be an arbitrary string.

$\text{len}(xy) = \text{len}(xwa) = \text{len}(xw) + 1$ (by definition of len)

Recursive rule says "every $a \in \Sigma$ " so we need to argue for every a .

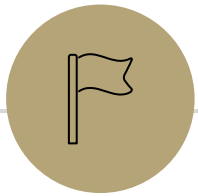
$= \text{len}(x) + \text{len}(w) + 1$ (by IH)

$= \text{len}(x) + \text{len}(wa)$ (by definition of len)

$P(y)$ is a for-all statement, introduce arbitrary variable to show for-all.

Therefore, $\text{len}(xy) = \text{len}(x) + \text{len}(y)$, as required.

We conclude that $P(y)$ holds for all strings y by the principle of induction. Unwrapping the definition of P , we get $\forall x \forall y \in \Sigma^* \text{len}(xy) = \text{len}(x) + \text{len}(y)$, as required.



A few last comments

What does the inductive step look like?

Here's a recursively-defined set:

Basis: $0 \in T$ and $5 \in T$

Recursive: If $x, y \in T$ then $x + y \in T$ and $x - y \in T$.

Let $P(x)$ be " $5|x$ "

What does the inductive step look like?

Well there's two recursive rules, so we have two things to show

Just the IS (you still need the other steps)

Let t be an arbitrary element of T not covered by the base case. By the exclusion rule $t = x + y$ or $t = x - y$ for $x, y \in T$.

Inductive hypothesis: Suppose $P(x)$ and $P(y)$ hold.

Case 1: $t = x + y$

By IH $5|x$ and $5|y$ so $5a = x$ and $5b = y$ for integers a, b .

Adding, we get $x + y = 5a + 5b = 5(a + b)$. Since a, b are integers, so is $a + b$, and $P(x + y)$, i.e. $P(t)$, holds.

Case 2: $t = x - y$

By IH $5|x$ and $5|y$ so $5a = x$ and $5b = y$ for integers a, b .

Subtracting, we get $x - y = 5a - 5b = 5(a - b)$. Since a, b are integers, so is $a - b$, and $P(x - y)$, i.e., $P(t)$, holds.

In all cases, we have $P(t)$. By the principle of induction, $P(x)$ holds for all $x \in T$.

If you don't have a recursively-defined set

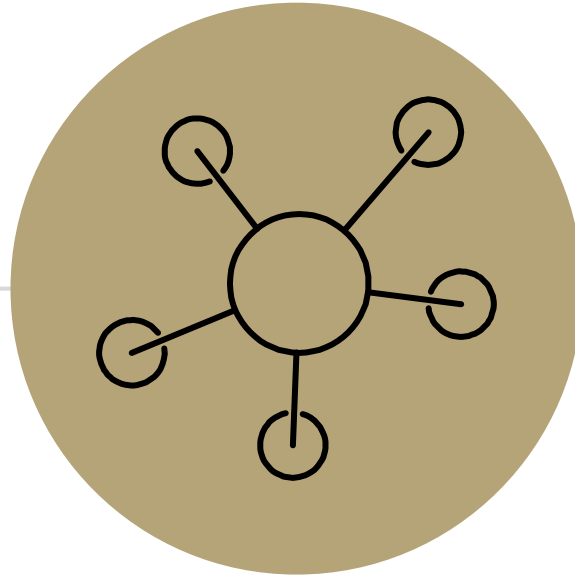
You won't do structural induction.

You can do weak or strong induction though.

For example, Let $P(n)$ be "for all elements of S of "size" n <something> is true"

To prove "for all $x \in S$ of size n ..." you need to start with "let x be an arbitrary element of size $k + 1$ in your IS.

You CAN'T start with size k and "build up" to an arbitrary element of size $k + 1$ it isn't arbitrary.



Part 3 of the course!

Course Outline

Symbolic Logic (training wheels)

Just make arguments in mechanical ways.

Set Theory/Number Theory (bike in your backyard)

Models of computation (biking in your neighborhood)

Still make and communicate rigorous arguments

But now with objects you haven't used before.

- A first taste of how we can argue rigorously about computers.

First up: regular expressions, context free grammars, automata – understand these “simpler computers”

Soon: what these simple computers can do

Then: what simple computers can't do.

Last week: A problem our computers cannot solve.



Extra Practice

Induction: Hats!

You have n people in a line ($n \geq 2$). Each of them wears either a **purple hat** or a **gold hat**. The person at the front of the line wears a purple hat. The person at the back of the line wears a gold hat.

Show that for every arrangement of the line satisfying the rule above, there is a person with a purple hat next to someone with a gold hat.

Yes, this is kinda obvious. I promise this is good induction practice.

Yes, you could argue this by contradiction. I promise this is good induction practice.

Induction: Hats!

Define $P(n)$ to be "in every line of n people with gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show $P(n)$ for all integers $n \geq 2$ by induction on n .

Base Case: $n = 2$

Inductive Hypothesis:

Inductive Step:

By the principle of induction, we have $P(n)$ for all $n \geq 2$

Induction: Hats!

Define $P(n)$ to be "in every line of n people with gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show $P(n)$ for all integers $n \geq 2$ by induction on n .

Base Case: $n = 2$ The line must be just a person with a purple hat and a person with a gold hat, who are next to each other.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step: Consider an arbitrary line with $k + 1$ people in purple and gold hats, with a gold hat at one end and a purple hat at the other.

Target: there is someone in a purple hat next to someone in a gold hat.

By the principle of induction, we have $P(n)$ for all $n \geq 2$

Induction: Hats!

Define $P(n)$ to be "in every line of n people with gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show $P(n)$ for all integers $n \geq 2$ by induction on n .

Base Case: $n = 2$ The line must be just a person with a purple hat and a person with a gold hat, who are next to each other.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step: Consider an arbitrary line with $k + 1$ people in purple and gold hats, with a gold hat at one end and a purple hat at the other.

Case 1: There is someone with a purple hat next to the person in the gold hat at one end. Then those people are the required adjacent opposite hats.

Case 2: There is a person with a gold hat next to the person in the gold hat at the end. Then the line from the second person to the end is length k , has a gold hat at one end and a purple hat at the other. Applying the inductive hypothesis, there is an adjacent, opposite-hat wearing pair.

In either case we have $P(k + 1)$.

By the principle of induction, we have $P(n)$ for all $n \geq 2$