Warm-up:
Show "if a^2 is even, then a is even."

Proof by Contradiction

CSE 311 Autumn 2024
Lecture 12

Trying a direct proof

$\forall a(\texttt{Even}(a^2) \rightarrow \texttt{Even}(a))$

Let a be an arbitrary integer and suppose that a^2 is even.

By definition of even, $a^2 = 2k$ for some integer k.

Taking the positive square-root of each side, we get $a = \sqrt{2k}$

....

Therefore a is even.

Taking a square root of a variable is tricky! It's hard to do algebra on.

Proving by contrapositive

$\forall a(\texttt{Even}(a^2) \rightarrow \texttt{Even}(a)) \equiv \forall a(\neg\texttt{Even}(a) \rightarrow \neg\texttt{Even}(a^2)) \equiv \forall a(\texttt{Odd}(a) \rightarrow \texttt{Odd}\,(a^2))$

We argue by contrapositive.

Let a be an arbitrary integer and suppose a is odd.

By definition of odd, $a = 2k + 1$ for some integer k.

Squaring both sides, we get $a^2 = (2k + 1)^2 = 4k^2 + 4k + 1$

Rearranging, we get $a^2 = 2(2k^2 + 2k) + 1$. Since k is an integer, $2k^2 + 2k$ is an integer, we thus get that a^2 meets the definition of odd (being 2 times an integer plus one), as required.

Since a was arbitrary, we have that for every odd a, that a^2 is also odd, which is the contrapositive of our original claim.

Signs you might want to use proof by contrapositive

1. The hypothesis of the implication you're proving has a "not" in it (that you think is making things difficult)

2. The target of the implication you're proving has an "or" or "not" in it.

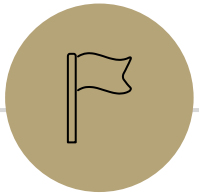
3. There's a step that is difficult forward, but easy backwards

   e.g., taking a square-root forward, squaring backwards.

4. You get halfway through the proof and you can't "get ahold of" what you're trying to show.

   e.g., you're working with a "not equal" instead of an "equals" or "every thing doesn't have this property" instead of "some thing does have that property"

All of these are reasons you **might** want contrapositive. Sometimes you just have to try and see what happens!



Proof by Contradiction

Proof By Contradiction

Suppose the negation of your claim.

Show that you can derive `False` (i.e. (¬claim) → `F`)

A correct proof shows that the implication is true.

So ¬claim must be `False`.

So claim must be `True`!

Proof By Contradiction Skeleton

Suppose, for the sake of contradiction $\neg p$

…

q

…

$\neg q$

But q and $\neg q$ is a contradiction! So we must have p.

Proof By Contradiction

Claim: $\sqrt{2}$ is irrational (i.e. not rational).

Proof:

Rational

A real number x is rational if (and only if) there exist integers p and q, with $q \neq 0$ such that $x = p/q$.

$$\texttt{Rational}(x) \coloneqq \exists p \exists q (\texttt{Integer}(p) \land \texttt{Integer}(q) \land (x = p/q) \land q \neq 0)$$

Proof By Contradiction

Claim: $\sqrt{2}$ is irrational (i.e. not rational).

Proof:

Suppose for the sake of contradiction that $\sqrt{2}$ is rational.

But [] is a contradiction!

We don't have a fixed target.
That can make this proof harder.

Proof By Contradiction

If a^2 is even then a is even.

Claim: $\sqrt{2}$ is irrational (i.e. not rational).

Proof:

Suppose for the sake of contradiction that $\sqrt{2}$ is rational.

By definition of rational, there are integers s, t such that $t \neq 0$ and $\sqrt{2} = s/t$. Without loss of generality, let s/t be in lowest terms (i.e., with no common factors greater than 1).

Fancy mathematician speak for
"It looks like I'm choosing more specific values, but it's ok for me to do that--- they're really still arbitrary"

That's a contradiction! We conclude $\sqrt{2}$ is irrational.

When can I say without loss of generality?

The claim you're trying to prove is fully general still. What you're doing looks like a new assumption but isn't. (Here: the variables are still arbitrary)

Here: we'd just divide p, q by their common factors (i.e., put the fraction in lowest-terms) and continue the proof.

Another common example:

Let x, y be integers; without loss of generality, assume $x \geq y$ (one of them must be bigger, just give the bigger one the name x).

Only use if your reader will immediately agree that you can still prove the claim! If you're worried, tell the reader how to get those values (here, define p, q as the reduced fraction, and continue with p, q as variables).

Proof By Contradiction

If a^2 is even then a is even.

Claim: $\sqrt{2}$ is irrational (i.e. not rational).

Proof:

Suppose for the sake of contradiction that $\sqrt{2}$ is rational.

By definition of rational, there are integers s, t such that $t \neq 0$ and $\sqrt{2} = s/t$. Without loss of generality, let s/t be in lowest terms (i.e., with no common factors greater than 1).

$$\sqrt{2} = \frac{s}{t}$$

$$2 = \frac{s^2}{t^2}$$

$2t^2 = s^2$ so s^2 is even.

That's a contradiction! We conclude $\sqrt{2}$ is irrational.

Proof By Contradiction

If a^2 is even then a is even.

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$$\sqrt{2} = \frac{s}{t}$$

$$2 = \frac{s^2}{t^2}$$

$2t^2 = s^2$ so s^2 is even. By the fact above, s is even, i.e. $s = 2k$ for some integer k. Squaring both sides $s^2 = 4k^2$

Substituting into our original equation, we have: $2t^2 = 4k^2$, i.e. $t^2 = 2k^2$.

So t^2 is even (by definition of even). Applying the fact above again, t is even.

But if both s and t are even, they have a common factor of 2. But we said the fraction was in lowest terms.

That's a contradiction! We conclude $\sqrt{2}$ is irrational.

Proof By Contradiction

How in the world did we know how to do that?

In real life…lots of attempts that didn't work.

Be very careful with proof by contradiction – without a clear target, you can easily end up in a loop of trying random things and getting nowhere.

What's the difference?

What's the difference between proof by contrapositive and proof by contradiction?

Show $p \to q$	Proof by contradiction	Proof by contrapositive
Starting Point	$\neg(p \to q) \equiv (p \wedge \neg q)$	$\neg q$
Target	Something false	$\neg p$

Show p	Proof by contradiction	Proof by contrapositive
Starting Point	$\neg p$	---
Target	Something false	---

Another Proof By Contradiction

Claim: There are infinitely many primes.

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Proof:

Suppose for the sake of contradiction, that there are only finitely many primes. Call them $p_1, p_2, \ldots, p_k$.

But [] is a contradiction! So there must be infinitely many primes.

Another Proof By Contradiction

Claim: There are infinitely many primes.

Proof:

Suppose for the sake of contradiction, that there are only finitely many primes. Call them $p_1, p_2, \ldots, p_k$.

Consider the number $q = p_1 \cdot p_2 \cdot \cdots \cdot p_k + 1$

Case 1: q is prime

Case 2: q is composite

But [] is a contradiction! So there must be infinitely many primes.

Another Proof By Contradiction

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Proof:

Suppose for the sake of contradiction, that there are only finitely many primes. Call them $p_1, p_2, \ldots, p_k$.

Consider the number $q = p_1 \cdot p_2 \cdot \cdots \cdot p_k + 1$

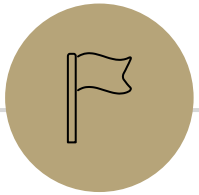
Case 1: q is prime

$q > p_i$ for all i. But every prime was supposed to be on the list $p_1, \ldots, p_k$. A contradiction!

Case 2: q is composite

Some prime on the list (say p_i) divides q. So $q\%p_i = 0$. and $(p_1 p_2 \cdots p_k + 1)\%p_i = 1$. But $q = (p_1 p_2 \cdots p_k + 1)$. That's a contradiction!

In either case we have a contradiction! So there must be infinitely many primes.



Extra Practice

Just the Skeleton

"For all integers x, if x^2 is even, then x is even."

Just the Skeleton

"For all integers x, if x^2 is even, then x is even."

Suppose for the sake of contradiction, there is an integer x, such that x^2 is even and x is odd.

…

[] is a contradiction, so for all integers x, if x^2 is even, then x is even.

Just the Skeleton

"There is not an integer k such that for all integers n, $k \geq n$.

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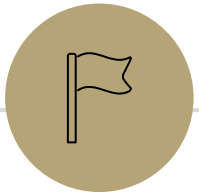
"There is not an integer k such that for all integers n, $k \geq n$.

Suppose, for the sake of contradiction, that there is an integer k such that for all integers n, $k \geq n$.

…

[] is a contradiction! So there is not an integer k such that for all integers n, $k \geq n$.



Small Techniques

Proof By Cases

If x is prime then x^2 is odd or $2|x$.

We need two different arguments – one for 2 and one for all the other primes…

Proof By Cases

Let x be an arbitrary prime number

We divide into two cases.

Case 1: x is even

If x is even then $x = 2k$ for some integer k, this is the definitions of $2|x$.

Case 2: x is odd

If x is odd, then $x = 2j + 1$ for some integer j. Squaring, we get
$x^2 = 4j^2 + 4j + 1 = 2(2j^2 + 2j) + 1.$
Since j is an integer $2j^2 + 2j$ is as well, so x^2 is odd by definition.

In either case, x met the condition of $2|x$ or x^2 is odd, so

Proof By Cases

Make it clear how you decide which case your in.

It should be obvious your cases are “exhaustive”

Reach the same conclusion in each of the cases, and you can say you’ve got that conclusion no matter what (outside the cases).

Advanced version: sometimes you end up arguing a certain case “can’t happen”

Two claims, two proof techniques

Suppose I claim that for all integers, if x is even then $8|x^2$.

That…doesn't look right.

How do you prove me wrong?

Want to show: $\exists x(Even(x) \wedge \neg[8|x\text{^}2])$

Consider $x = 6$. Then x is even (since $6 = 3 \cdot 2$), but 8 does not divide 36 (as $36\%8 = 4$).

Proof By [Counter]Example

To prove an existential statement (or disprove a universal statement), provide an example, and demonstrate that it is the needed example.

You don't have to explain where it came from! (In fact, you **shouldn't)**

Computer scientists and mathematicians like to keep an air of mystery around our proofs.

(or more charitably, we want to focus on just enough to believe the claim)

Skeleton of an Exists Proof

To show $\exists x(P(x))$

Consider x =[the value that will work]

[Show that x does cause $P(x)$ to be true.]

So [value] is the desired x.

You'll probably need some "scratch work" to determine what to set x to. That might not end up in the final proof!